1. Grading rubric: 16 points are possible. On each part A-H, assign 2 points for a correct answer, 1 point for PUNT, and 0 points for an incorrect answer. No justification is needed.

A. About asymptotics: $\forall f\ \forall g\ \ (f = \mathcal{O}(g) \Leftrightarrow \exists N\ \forall n \geq N\ f(n) \leq g(n))$.

FALSE. (This is the definition of $\mathcal{O}$-notation, but omitting the multiplicative constant c.)

B. About propositional logic: For any proposition p, there exists a proposition q in conjunctive normal form such that $p \equiv q$.

TRUE. (This is Theorem 4.2, on page 323 of our textbook.)

C. About integers: $\forall a\ \forall b\ \forall c\ (a|bc \Rightarrow (a|b \vee a|c))$.

FALSE. (This is true only for integers a that are $\pm$prime or ± 1.)

D. About people: Let $m(x, y)$ be "x is the mother of y". Then $\forall x\ \exists y\ m(x, y)$.

FALSE. (This says that everyone is somebody's mother.)

E. The truth table for the compound proposition $(p \Rightarrow q) \vee (r \Rightarrow \neg q)$ has exactly 6 true rows.

FALSE. (It has 8 true rows.)

F. $\forall x\ \forall y\ p(x, y)$ is logically equivalent to $\forall y\ \forall x\ p(x, y)$ for all predicates $p(x, y)$ and all domains.

TRUE. (You can reorder quantifiers of the same type.)

G. $\neg\forall x\ (p(x) \wedge q(x)) \equiv \exists x\ (\neg p(x) \wedge \neg q(x))$ for all predicates $p(x)$ and $q(x)$ and all domains.

FALSE. (This is like DeMorgan's laws, but one of the $\wedge$s should be a $\vee$.)

H. Let p be the statement "Elvis is 6 years old" and q the statement "Elvis enjoys Disney World." Then the statement "Elvis enjoys Disney World only if Elvis is 6 years old" is $p \Rightarrow q$.

FALSE. (It's $q \Rightarrow p$.)

2. Grading rubric: 12 points are possible. Assign 4 points for recognizing that odd means $n = 2k + 1$. Assign 4 points for the key insight of $(k + 1)^2 - k^2 = 2k + 1$. Assign up to 4 points for writing it up coherently.

Let n be an odd integer. Then there exists an integer k such that $n = 2k + 1$. Notice that $(k + 1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n$. Thus n is the difference of the squares $(k + 1)^2$ and k^2.

3A. Grading rubric: 6 points are possible. Assign 3 points for the key insight that the second player wants to leave a multiple of 3 squirrels. Assign up to 3 points for writing it up coherently.

An integer n is divisible by 3 if and only if there exists an integer k such that $n = 3k$. We proceed by induction on k.

Base case: $k = 1$: In this case, $n = 3$. If the first player takes 1 squirrel, then the second player takes 2 squirrels and wins. If the first player takes 2 squirrels, then the second player takes 1 squirrel and wins. So the second player always wins.

Inductive case: Assume that $k \geq 2$, and that the second player always wins the game with $3k$ squirrels. We wish to show that the second player always wins the game with $3(k + 1) = 3k + 3$

squirrels. If the second player responds to the first player's first move as described in the base case, then $3k$ squirrels remain, and it is again the first player's move. By the inductive hypothesis, the second player wins the game.

Therefore, by the principle of mathematical induction, the second player always wins when n is a multiple of 3.

3B. Grading Rubric: 6 points are possible. Assign 3 points for the key insight that the first player wants to leave a multiple of 3 squirrels. Assign up to 3 points for writing it up coherently. Notice that the proof below uses the proof above for its heavy lifting.

If n is not a multiple of 3, then it is either 1 more or 2 more than a multiple of 3. In the former case, the first player should take 1 squirrel; in the latter case, the first player should take 2 squirrels. In either case, a multiple of 3 squirrels remains. The game then proceeds as in Part A, with the roles of the two players switched. The first player will always win, by responding to the second player in a way that always leaves a multiple of 3 squirrels.

4A. Grading rubric: 4 points are possible. The definition would ideally mimic the one for binary trees in our class and textbook. If it doesn't, then make sure that it handles the smallest cases (empty trees, single-node trees) as well as larger cases.

A B-ary tree is either (A) empty, or (B) a root node attached to B "children", each of which is a B-ary tree.

4B. Grading rubric: 8 points are possible. Assign 4 points if the crucial algebra is present. Assign up to 4 points for writing the proof coherently.

We proceed by structural induction, on the recursive definition of B-ary trees.

Base case: The tree is empty. So $n = 0$, $h = -1$, and $0 \leq B^{-1+1} - 1 = 0$ is true, no matter which $B \geq 2$ is being used.

Inductive case: The tree is a root node r attached to children $T_1, \ldots, T_B$, each of which is a B-ary tree. Let h_k be the height of T_k, and n_k the number of nodes in T_k. We assume, for the sake of structural induction, that $n_k \leq B^{h_k+1} - 1$ for all k. Let m be the maximum of the heights h_k. Then $h = 1 + m$, and

$$\begin{aligned}
n &= 1 + \sum_{k=1}^{B} n_k \\
&\le 1 + \sum_{k=1}^{B} \left(B^{h_k+1} - 1\right) && \text{(by the inductive hypothesis)} \\
&\le 1 + \sum_{k=1}^{B} \left(B^{m+1} - 1\right) && \text{(because } h_k \le m \text{ for all } k\text{)} \\
&= 1 + B\left(B^{h} - 1\right) && \text{(because } h = m + 1\text{)} \\
&= B^{h+1} - B + 1 \\
&\le B^{h+1} - 1 && \text{(because } B \ge 2\text{).}
\end{aligned}$$

This completes the inductive case and the proof by structural induction.