INTRODUCTION TO FLUID DYNAMICS

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Abstract. This brief paper derives Euler's equations for an inviscid fluid, summarizes the Cauchy momentum equation, derives the Navier-Stokes equation from that, and then talks about finite difference method approaches to solutions.

Typical texts for this material are apparently Acheson, *Elementary Fluid Dynamics* and Landau and Lifschitz, *Fluid Mechanics*.

1. Basic Definitions

We describe a fluid flow in three-dimensional space $\mathbb{R}^3$ as a vector field representing the velocity at all locations in the fluid. Concretely, then, a *fluid flow* is a function

$$\vec{v} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$$

that assigns to each point $(t, \vec{x})$ in spacetime a velocity $\vec{v}(t, \vec{x})$ in space. In the special situation where $\vec{v}$ does not depend on t we say that the flow is *steady*.

A *trajectory* or *particle path* is a curve $\vec{x} : \mathbb{R} \to \mathbb{R}^3$ such that for all $t \in \mathbb{R}$,

$$\frac{d}{dt}\vec{x}(t) = \vec{v}(t, \vec{x}(t)).$$

Fix a $t_0 \in \mathbb{R}$; a *streamline* at time t_0 is a curve $\vec{x} : \mathbb{R} \to \mathbb{R}^3$ such that for all $t \in \mathbb{R}$,

$$\frac{d}{dt}\vec{x}(t) = \vec{v}(t_0, \vec{x}(t)).$$

In the special case of steady flow the streamlines are constant across times t_0 and any trajectory is a streamline. In non-steady flows, particle paths need not be streamlines.

Consider the 2-dimensional example $\vec{v} = [-\sin t \quad \cos t]^\top$. At $t_0 = 0$ the velocities all point up and the streamlines are vertical straight lines. At $t_0 = \pi/2$ the velocities all point left and the streamlines are horizontal straight lines. Any trajectory is of the form $\vec{x} = [\cos t + C_1 \quad \sin t + C_2]^\top$; this traces out a radius-1 circle centered at $[C_1 \quad C_2]^\top$. Indeed, *all* radius-1 circles in the plane arise as trajectories. These circles cross each other at many (in fact, all) points. If you find it counterintuitive that distinct trajectories can pass through a single point, remember that they do so at different times.

2. Acceleration

Let $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be some scalar field (such as temperature). Then $\partial f/\partial t$ is the rate of change of f at some fixed point in space. If we precompose f with a trajectory $\vec{x}$, then the chain rule gives us the rate of change of f with respect to time along that curve:

$$\begin{aligned}\frac{D}{Dt}f &:= \frac{d}{dt}f(t,x(t),y(t),z(t)) \\ &= \frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= \left(\frac{\partial}{\partial t}+\frac{dx}{dt}\frac{\partial}{\partial x}+\frac{dy}{dt}\frac{\partial}{\partial y}+\frac{dz}{dt}\frac{\partial}{\partial z}\right)f \\ &= \left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)f.\end{aligned}$$

Intuitively, if $\vec{x}$ describes the trajectory of a small sensor for the quantity f (such as a thermometer), then Df/Dt gives the rate of change of the output of the sensor with respect to time. The $\partial f/\partial t$ term arises because f varies with time. The $\vec{v}\cdot\nabla f$ term arises because f is being measured at varying points in space.

If we apply this idea to each component function of $\vec{v}$, then we obtain an acceleration (or force per unit mass) vector field

$$\vec{a}(t,x) := \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}.$$

That is, for any spacetime point $(t,\vec{x})$, the vector $\vec{a}(t,\vec{x})$ is the acceleration of the particle whose trajectory happens to pass through $\vec{x}$ at time t.

Let's check that it agrees with our usual notion of acceleration. Suppose that a curve $\vec{x}$ describes the trajectory of a particle. The acceleration should be $\frac{d}{dt}\frac{d}{dt}\vec{x}$. By the definition of trajectory,

$$\frac{d}{dt}\frac{d}{dt}\vec{x} = \frac{d}{dt}\vec{v}(t,\vec{x}(t)).$$

The right-hand side is precisely $D\vec{v}/Dt$.

Returning to our 2-dimensional example $\vec{v} = [-\sin t \quad \cos t]^\top$, we have $\vec{a} = [-\cos t \quad -\sin t]^\top$. Notice that $\vec{v}\cdot\vec{a} = 0$. This is the well-known fact that in constant-speed circular motion the centripetal acceleration is perpendicular to the velocity. (In fact, the acceleration of any constant-speed trajectory is perpendicular to its velocity.)

3. Ideal Fluids

An *ideal fluid* is one of constant density ρ, such that for any surface within the fluid the only stresses on the surface are normal. That is, there exists a scalar field $p : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, called the *pressure*, such that for any surface element ΔS with outward-pointing unit normal vector $\vec{n}$, the force exerted by the fluid inside ΔS on the fluid outside ΔS is $p\vec{n}\,\Delta S$.

The constant density condition implies that the fluid is *incompressible*, meaning $\nabla\cdot\vec{v} = 0$, as follows. For any region of space R, the rate of flow of mass out of the region is

$$\iint_{\partial R} \rho\vec{v}\cdot\vec{n}\,dS = \iiint_R \nabla\cdot(\rho\vec{v})\,dV$$

(by the divergence theorem). Because the density ρ is constant, there can be no change in the mass of fluid in R; the triple integral is zero. Because this holds for all regions R, the integrand $\nabla \cdot (\rho \vec{v}) = \rho \nabla \cdot \vec{v}$ must be zero, so the fluid is incompressible.

[Warning: The texts assume divergence zero in addition to constant density, instead of deriving the former from the latter. But I don't see an error here.]

4. Euler's Equation

In an ideal fluid with pressure p, the total force exerted on a small chunk E of fluid by the fluid around it is the vector $\iint_{\partial E} -p\vec{n}\, dS$. Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the standard basis vectors for $\mathbb{R}^3$. Then for $i = 1, 2, 3$,

$$\begin{aligned} \left(\iint_{\partial E} -p\vec{n}\, dS\right) \cdot \vec{e}_i &= \iint_{\partial E} -p\vec{e}_i \cdot \vec{n}\, dS \\ &= \iiint_E \nabla \cdot (-p\vec{e}_i)\, dV \\ &= \iiint_E -\nabla p \cdot \vec{e}_i\, dV \\ &= \left(\iiint_E -\nabla p\, dV\right) \cdot \vec{e}_i. \end{aligned}$$

Therefore the force exerted on the chunk by the surrounding fluid is

$$\iint_{\partial E} -p\vec{n}\, dS = \iiint_E -\nabla p\, dV.$$

Working infinitesimally, suppose that the chunk has volume ΔV. By the preceding equation, the total force on the chunk is $-(\nabla p)\Delta V$. Its mass is $\rho \Delta V$, and its acceleration is $\frac{D\vec{v}}{Dt}$. By Newton's second law of motion,

$$-(\nabla p)\Delta V = \rho \Delta V \cdot \frac{D\vec{v}}{Dt}.$$

Dividing through by $\rho \Delta V$ and expanding $\frac{D\vec{v}}{Dt}$ we obtain *Euler's equation*

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p.$$

5. Cauchy momentum equation

In a more general fluid, there may be tangential stresses, which arise from the friction of particles sliding past each other, in addition to normal stresses. The *Cauchy momentum equation* is Euler's equation with an extra term to incorporate these new stresses:

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\nabla p + \nabla \cdot T.$$

Here, T is a symmetric 3×3 matrix

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix},$$

and $\nabla \cdot T$ is the vector

$$\nabla \cdot T = \begin{bmatrix} \nabla \cdot \langle T_{11}, T_{12}, T_{13} \rangle \\ \nabla \cdot \langle T_{21}, T_{22}, T_{23} \rangle \\ \nabla \cdot \langle T_{31}, T_{32}, T_{33} \rangle \end{bmatrix}.$$

We do not derive this equation in this paper.

The Cauchy momentum equation is a highly general relationship between *strain* (particle movement, on the left side) and *stress* (force per area, on the right side). It holds for a wide variety of materials, including various kinds of fluids and solids. To specialize the theory to one particular kind of material, you introduce another stress-strain relationship, called the *constitutive equation* or *rheology*.

As a trivial example, plugging $T = 0$ (where 0 is the 3×3 matrix of zeros) into the Cauchy momentum equation results in Euler's equation. So ideal fluids are materials defined by the constitutive equation $T = 0$.

6. Navier-Stokes equation

A *Newtonian fluid* is one that obeys the Cauchy momentum equation (of course), the incompressibility equation $\nabla \cdot \vec{v} = 0$, and the constitutive equations

$$T_{ij} = \frac{\mu}{\rho} \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),$$

where μ is a constant called *viscosity* and ρ is the constant density. Roughly, this constitutive equation says that stress is proportional to the gradient of velocity. The proportionality constant μ/ρ measures how "thick" the fluid is. Typical values for μ/ρ are 0.01 (water), 0.15 (air), 1.0 (olive oil), 18 (glycerine), and 1,200 (syrup).

The *Navier-Stokes equation* is Cauchy's momentum equation with this constitutive equation plugged in:

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \Delta \vec{v}.$$

The theory behind the Navier-Stokes equation is notoriously difficult. When a mathematician encounters a differential equation such as this, her first questions are: "Do solutions exist? Are the solutions smooth? Is there a unique solution?" For Navier-Stokes, the answers are not known. In 2000, the Clay Mathematics Institute announced a \$1 million prize for finding them.

7. Finite difference method

!!under construction...