A.

$\vec{v} + \vec{w} = \langle 0, 2, 3 \rangle$.

$7\vec{v} = \langle -14, 21, 7 \rangle$.

$\vec{v} \cdot \vec{w} = -4 + -3 + 2 = -5$.

$\text{proj}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{-5}{9} \langle 2, -1, 2 \rangle = \langle -10/9, 5/9, -10/9 \rangle$.

$\vec{v} \times \vec{w} = ... = \langle 7, 6, -4 \rangle$.

B. [By the way, this is a simplified version of Section 14.3 Exercise 80.] First,

$$f_x = 3x^2 + 2bxy + cy^2 \quad \Rightarrow \quad f_{xx} = 6x + 2by,$$

and

$$f_y = bx^2 + 2cxy + 3y^2 \quad \Rightarrow \quad f_{yy} = 2cx + 6y.$$

Therefore

$$0 = f_{xx} + f_{yy} = (2c + 6)x + (2b + 6)y.$$

Because this equation holds for all x and y, it must be true that $2c + 6 = 0$ and $2b + 6 = 0$. In other words, $b = c = -3$.

C. Because $\vec{r}(t)$ is parametrized by arc length,

$$\begin{aligned} & 1 &=& \; ||\vec{r}\,'(t)|| \\ \Rightarrow \; & 1 &=& \; ||\vec{r}\,'(t)||^2 \\ & &=& \; \vec{r}\,'(t) \cdot \vec{r}\,'(t) \\ \Rightarrow \; & 0 &=& \; \vec{r}\,''(t) \cdot \vec{r}\,'(t) + \vec{r}\,'(t) \cdot \vec{r}\,''(t) \\ & &=& \; 2\vec{r}\,'(t) \cdot \vec{r}\,''(t). \end{aligned}$$

Therefore $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$ are perpendicular, and

$$||\vec{r}\,'(t) \times \vec{r}\,''(t)|| = ||\vec{r}\,'(t)|| ||\vec{r}\,''(t)|| = ||\vec{r}\,''(t)||.$$

The expression for $\kappa(t)$ then simplifies down to

$$\kappa(t) = ||\vec{r}\,''(t)|| = ||d\vec{r}\,'/dt|| = ||d\vec{r}\,'/ds||,$$

which is the usual definition for κ.

D. By the definition of $\vec{T}$ and the assumption that $\vec{r}$ is parametrized by arc length,

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} = \vec{r}\,'(t).$$

Differentiating both sides of that equation with respect to $s = t$ gives

$$d\vec{T}/ds = \vec{T}'(t) = \vec{r}''(t).$$

Then, by the definition of the normal $\vec{N}$,

$$\begin{aligned} \vec{N} &= \frac{\vec{T}\,'(t)}{||\vec{T}\,'(t)||} \\ &= \frac{d\vec{T}/ds}{||\vec{r}''(t)||} \\ &= \frac{d\vec{T}/ds}{\kappa} \\ \Rightarrow \quad d\vec{T}/ds &= \kappa\vec{N}(t). \end{aligned}$$

So we have derived the first Frenet-Serret equation.

E1. [By the way, this is Section 14.2 Exercise 15.] First, plugging in $y = mx$ gives

$$f(x, mx) = \frac{x^3 + m^3x^3}{xm^2x^2} = \frac{1 + m^3}{m^2}.$$

Therefore the limit along the line $y = mx$ is

$$\lim_{x \to 0} \frac{1 + m^3}{m^2} = \frac{1 + m^3}{m^2}.$$

Notice that different slopes m produce different limits.

E2. The apparent limit of $f(x, y)$ as $(x, y) \to (0, 0)$ depends on how (x, y) approaches $(0, 0)$. Therefore the limit does not actually exist.

F1. [By the way, this is an alternate form of Section 11.3 Exercise 55, which was assigned as homework.] Well,

$$\begin{aligned} r^2 &= \cos^2\theta - \sin^2\theta \\ \Rightarrow \quad r^4 &= (r\cos\theta)^2 - (r\sin\theta)^2 \\ \Rightarrow \quad (x^2 + y^2)^2 &= x^2 - y^2. \end{aligned}$$

F2. Using y' as shorthand for dy/dx, we compute

$$\begin{aligned} (x^2 + y^2)^2 &= x^2 - y^2 \\ \Rightarrow 2(x^2 + y^2)(2x + 2yy') &= 2x - 2yy' \\ \Rightarrow \left(y + 2y(x^2 + y^2)\right) y' &= x - 2x(x^2 + y^2) \\ \Rightarrow y' &= \frac{x}{y} \cdot \frac{1 - 2(x^2 + y^2)}{1 + 2(x^2 + y^2)}. \end{aligned}$$

F3. We are looking for points where $x(1 - 2(x^2 + y^2)) = 0$. There are two possibilities: $x = 0$ or $x^2 + y^2 = 1/2$.

Plugging $x = 0$ into the answer for Question F1 implies that $y = 0$. However, the derivative computed in Question F2 is undefined at $(0, 0)$, leading us to suspect, without certainty, that the tangent line there is not horizontal.

Plugging $x^2 + y^2 = 1/2$ into the answer for Question F1 implies that $x^2 - y^2 = 1/4$. Together these equations imply that $x^2 = 3/8$ and $y^2 = 1/8$. So we conclude that $x = \pm\sqrt{3/8}$ and $y = \pm\sqrt{1/8}$.

[The preceding answer is sufficient for a timed exam without a graphing aid. In practice, one would graph the function. The graph confirms that the four points described by $x = \pm\sqrt{3/8}$ and $y = \pm\sqrt{1/8}$ have horizontal tangent lines. The graph also shows that $(0, 0)$ is a point where the curve crosses itself and thus does not have a well-defined tangent line. One can get some insight without graphing, too. Evaluate the limit of y' as $(x, y) \to (0, 0)$ along the curve. Use polar coordinates. The limit is $\cos\theta/\sin\theta$. For the tangent line to be horizontal, this limit should be 0. But then $\cos\theta$ approaches 0 along the curve, which means that the curve approaches the origin vertically, and therefore the tangent line cannot be horizontal.]

G1. [By the way, this is nearly identical to an example that we discussed in class. We showed a contour plot, constructed the surface out of Play-Doh, and graphed the surface in Mathematica. I will omit the plot in these solutions.]

G2. Briefly, $\nabla f = \langle 2x \sin y, x^2 \cos y\rangle$.

G3. The function is differentiable, so

$$D_{\vec{v}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{v} = \langle 2\sin 2, \cos 2\rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle = \langle -\sqrt{2}\sin 2, (\cos 2)/\sqrt{2}\rangle.$$