A1. Notice that $\vec{n} \times \vec{m}$ is perpendicular to both $\vec{n}$ and $\vec{m}$, and therefore lies in both planes, and therefore lies in the line where they intersect. Therefore the line can be parametrized as $\vec{r}(t) = \vec{s} + t(\vec{n} \times \vec{m})$.

A2. It will not work when $\vec{n}$ and $\vec{m}$ are scalar multiples of each other. For then the cross product will be $\vec{0}$ and the parametrized "line" will not be a line. Geometrically, this happens exactly when P and Q are identical.

B1. The potential function is $f = xe^{yz} + c$, where c is an arbitrary constant. [Check that $\frac{\partial f}{\partial x} = F_1$, etc.]

B2. By the fundamental theorem of calculus for line integrals,

$$\int_C \vec{F} \cdot d\vec{s} = \int_C (\nabla f) \cdot d\vec{s} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(\cos 1, 1, \log 2) - f(1, 0, 0) = 2(\cos 1) - 1.$$

C1. [By the way, this is Section 14.7 Exercise 12.] The gradient of f is $\nabla f = \langle 3x^2 - 6, 4y^3 - 4y \rangle$. It is never undefined. It equals $\vec{0}$ when

$$x^2 = 2, \quad y^3 = y.$$

The solutions are $x = \pm\sqrt{2}$ and $y = -1, 0, 1$. So there are six critical points.

C2. The second derivatives are

$$f_{xx} = 6x, \quad f_{yy} = 12y^2 - 4, \quad f_{xy} = f_{yx} = 0.$$

The discriminant is

$$f_{xx}f_{yy} - f_{xy}f_{yx} = 6x(12y^2 - 4).$$

At the critical point $(x, y) = (\sqrt{2}, 0)$, the discriminant is negative, so the point is a saddle point.

D1. [By the way, this is Section 17.2 Exercise 10.] We compute

$$\operatorname{curl} \vec{G} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} 2y \\ e^z \\ -\arctan x \end{bmatrix} = \begin{bmatrix} 0 - e^z \\ 0 - -\frac{1}{1+x^2} \\ 0 - 2 \end{bmatrix} = \vec{F}.$$

D2. By Stokes' theorem,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\operatorname{curl} \vec{G}) \cdot d\vec{S} = \int_{\partial S} \vec{G} \cdot d\vec{s}.$$

Parametrize ∂S by $\vec{r}(t) = (2\cos t, 2\sin t, 0)$, for $0 \le t \le 2\pi$. Its orientation is compatible with the upward-pointing normals on S. Then

$$\begin{aligned}
\int_{\partial S} \vec{G} \cdot d\vec{s} &= \int_0^{2\pi} \vec{G}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt \\
&= \int_0^{2\pi} \langle 4\sin t, e^0, -\arctan(2\cos t)\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\; dt \\
&= \int_0^{2\pi} -8\sin^2 t + 2\cos t\; dt \\
&= [-4t + 2\sin(2t) + 2\sin t]_0^{2\pi} \\
&= (-8\pi + 0 + 0) - (0 + 0 + 0) \\
&= -8\pi.
\end{aligned}$$

E1. [By the way, this is Section 15.3 Exercise 15. I'll omit the drawing in these typed solutions.]

E2. Based on the drawing above, we compute the iterated integral

$$\begin{aligned}
\iiint_W f(x,y,z)\; dV &= \int_0^1 \int_0^x \int_0^{\sqrt{9-x^2-y^2}} z\; dz\; dy\; dx \\
&= \int_0^1 \int_0^x \left[z^2/2\right]_0^{\sqrt{9-x^2-y^2}}\; dy\; dx \\
&= \frac{1}{2}\int_0^1 \int_0^x 9 - x^2 - y^2\; dy\; dx \\
&= \frac{1}{2}\int_0^1 \left[9y - x^2y - y^3/3\right]_0^x\; dx \\
&= \frac{1}{2}\int_0^1 9x - 4x^3/3\; dx \\
&= \frac{1}{2}[9x^2/2 - x^4/3]_0^1 \\
&= 9/4 - 1/6 \\
&= 25/12.
\end{aligned}$$

F. [By the way, this is Section 14.8 Example 1.] We wish to optimize $f(x,y) = 2x + 5y$ subject to $g(x,y) = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$. Proceeding by Lagrange multipliers, we compute $\nabla f = \langle 2, 5\rangle$ and $\nabla g = \langle x/8, 2y/9\rangle$. We arrive at a system of three equations in three unknowns:

$$\begin{aligned}
2 &= \lambda x/8, \\
5 &= 2\lambda y/9, \\
1 &= \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2.
\end{aligned}$$

Solving for λ in the first two equations yields $\lambda = 16/x = 45/(2y)$, which implies that $y = 45x/32$. Plugging this equation into the constraint produces $1 = (16^2 + 15^2)x^2/32^2$, which implies that

$$x = \pm\frac{32}{\sqrt{16^2 + 15^2}},$$

which implies that

$$y = \frac{45}{32}x = \pm\frac{45}{\sqrt{16^2 + 15^2}}.$$

So there are two points of concern: one with x and y positive, and the other with x and y negative. Because $f(x, y)$ increases with both x and y, it is greater at the positive solution than at the negative solution. Hence the former is the maximum (with value $(2\cdot 32+5\cdot 45)/\sqrt{16^2 + 15^2}$) and the latter the minimum (with opposite value).

G. [By the way, I often mention this concept and prove this result during the course, but this term I did not.] We just compute it out:

$$\mathrm{div}(\mathrm{grad}\ f) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f_x, f_y, f_z\rangle = f_{xx} + f_{yy} + f_{zz} = \Delta f.$$

H. [By the way, this was one of the study questions mentioned on the last day of class.] Because f is a scalar field, $\vec{F}$ is a vector field, and $\mathrm{div}(f\vec{F})$ is a scalar field, the rule is probably $\mathrm{div}(f\vec{F}) = \nabla f \cdot \vec{F} + f\mathrm{div}\vec{F}$.