A. For the ratio test, we compute

$$\left|\frac{(-1)^{n+1}(2x)^{2n+3}(2n+1)}{(2n+3)(-1)^n(2x)^{2n+1}}\right| = \left|(2x)^2\frac{2n+1}{2n+3}\right| = 4x^2\frac{2+1/n}{2+3/n}.$$

As $n \to \infty$, this expression limits to $4x^2$. So the series converges where $4x^2 < 1$ and diverges where $4x^2 > 1$. We must specifically check where $4x^2 = 1$, which is where $x = \pm\frac{1}{2}$. At $x = \pm\frac{1}{2}$ we have the series $\pm\sum \frac{(-1)^n}{2n+1}$, which converges by the alternating series test. [You fill in the details.] So the interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

B. For the root test, we compute

$$\left|\left(\frac{cn-1}{n+12}\right)^n\right|^{1/n} = \frac{|cn-1|}{n+12} = \frac{|c-1/n|}{1+12/n} \to |c|.$$

So the series should converge (absolutely) as long as $|c| < 1$. So $c = \frac{1}{2}$ works, for example.

C. It is easier to analyze

$$\begin{aligned}\lim_{n\to\infty} \log a_n &= \lim_{n\to\infty} \log\left(n^{1/n}\right) \\ &= \lim_{n\to\infty} \frac{\log n}{n} \\ &= \lim_{x\to\infty} \frac{\log x}{x} \\ &= \lim_{x\to\infty} \frac{1}{x} \\ &= 0.\end{aligned}$$

(The fourth equality follows by L'Hopital's rule.) Therefore $a_n$ converges to $e^0 = 1$.

D. By the integral test, the series converges if and only if the integral $\int_2^\infty \frac{1}{x \log x}\,dx$ converges. The integral is

$$\begin{aligned}\lim_{b\to\infty} \int_2^b x^{-1}(\log x)^{-1}\,dx &= \lim_{b\to\infty} [\log\log x]_2^b \\ &= \lim_{b\to\infty} \log\log b - \log\log 2 \\ &= \infty.\end{aligned}$$

So the series diverges. [Bonus problem: For which values of $q$ does $\sum \frac{1}{n^q \log n}$ converge?]

E.A. It's an alternating series. The absolute value of the $n$th term is $\sin^2(1/n)/n^{9/2}$, which is strictly decreasing. The limit of the terms is 0, because the numerator is between 0 and 1 and the denominator goes to infinity. So, by the alternating series test, the series converges. [Bonus: The series converges absolutely.]

E.B. The error in truncating after the 10th term is no larger than the absolute value of the 11th term, which is $\sin(1/11)/11^{9/2}$.