A. [This is Section 17.4 #3. Here is the "usual" strategy for solving the problem.] Suppose that $y = \sum_{n=0}^{\infty} c_n x^n$ for some as-yet-unknown coefficients c_n. Then

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n = c_1 + 2c_2 x + \sum_{n=2}^{\infty} c_{n+1}(n+1)x^n.$$

Meanwhile,

$$x^2 y = \sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{n=0}^{\infty} c_{n-2} x^n.$$

The power series for y' and $x^2 y$ must match term-by-term. From the x^0 terms we have $c_1 = 0$. From the x_1 terms we have $2c_2 = 0$. Then, for all $n \geq 2$, we have $c_{n+1}(n+1) = c_{n-2}$, which we rewrite as

$$c_{n+1} = \frac{c_{n-2}}{n+1}.$$

Therefore c_0 is arbitrary, $c_1 = 0$, and $c_2 = 0$. We get

$$\begin{aligned} c_3 &= \frac{c_0}{3}, \\ c_6 &= \frac{c_3}{6} = \frac{c_0}{6 \cdot 3}, \\ c_9 &= \frac{c_6}{9} = \frac{c_0}{9 \cdot 6 \cdot 3}, \end{aligned}$$

and so on. The pattern is

$$c_{3n} = \frac{c_0}{3n \cdot 3(n-1) \cdot \cdots \cdot 3} = \frac{c_0}{n! 3^n}.$$

All other coefficients in the power series are 0. So the answer is

$$y = \sum_{n=0} \frac{c_0}{n! 3^n} x^{3n} = c_0 \sum_{n=0} \frac{1}{n!} \left(\frac{x^3}{3} \right)^n = c_0 e^{x^3/3}.$$

[Here is a second solution — the "backwards" way.] By examining the differential equation we can guess that $y = ce^{x^3/3}$. Then we can reconstruct a power series by plugging $x^3/3$ into the power series for e^x:

$$y = ce^{x^3/3} = c \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^3}{3} \right)^n = \sum_{n=0}^{\infty} \frac{c}{n! 3^n} x^{3n}.$$

B. [I recommend that you draw a picture.] The problem is probably easiest in spherical coordinates, where E is described by $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi/2$, $0 \leq \rho \leq 1$. So the integral is

$$\begin{aligned}
\iiint_E z\, dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho \cos(\phi) \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/2} \cos\phi \sin\phi\, d\phi \cdot \int_0^1 \rho^3\, d\rho \\
&= 2\pi \cdot \frac{1}{2} \left[\sin^2 \phi\right]_0^{\pi/2} \cdot \frac{1}{4} \left[\rho^4\right]_0^1 \\
&= \frac{\pi}{4}(1-0)(1-0) \\
&= \frac{\pi}{4}.
\end{aligned}$$

C. The series converges, because

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n}-1} &\leq \sum_{n=2}^{\infty} \frac{1}{\frac{1}{2} n^{3/2}} \\
&= 2 \sum_{n=2}^{\infty} n^{-3/2},
\end{aligned}$$

which is a p-series with $p = -3/2 < -1$.

D.A. Let $\vec{r}(t)$ be any curve such that $|\vec{r}(0)| \approx \infty$ and $|\vec{r}(1)| = 1$. The work done by the force field is

$$\begin{aligned}
\int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt &= f(\vec{r}(1)) - f(\vec{r}(0)) \\
&\approx -\frac{c}{1} - 0 \\
&= -c,
\end{aligned}$$

by the fundamental theorem of calculus for line integrals. So the work that *we* must do against the force field is c.

D.B. Let $\vec{r}(t)$ be any curve such that $|\vec{r}(0)| = 1$ and $|\vec{r}(1)| = 0$. The work done by the force field is

$$\begin{aligned}
\int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt &= f(\vec{r}(1)) - f(\vec{r}(0)) \\
&\approx -\infty - -c \\
&= -\infty.
\end{aligned}$$

So we must do infinite work against the force field. [This argument can be made rigorous using limits.]

E.A. [This is identical to a problem from Exam B.] It is easier to analyze

$$\begin{aligned}
\lim_{n\to\infty} \log a_n &= \lim_{n\to\infty} \log\left(n^{1/n}\right) \\
&= \lim_{n\to\infty} \frac{\log n}{n} \\
&= \lim_{x\to\infty} \frac{\log x}{x} \\
&= \lim_{x\to\infty} \frac{1}{x} \\
&= 0.
\end{aligned}$$

(The fourth equality follows by L'Hopital's rule.) Therefore a_n converges to $e^0 = 1$.

E.B. Because a_n does not go to 0, we know immediately that the series diverges.

F. [I recommend that you draw a picture.] The curve is the circle of radius $\sqrt{8} = 2\sqrt{2}$ in the $z = 8$ plane, centered on the z-axis. So a parametrization is $\vec{r}(t) = (2\sqrt{2}\cos t, 2\sqrt{2}\sin t, 8)$, where t varies over $[0, 2\pi]$.

G. After a little work, we find that $f^{(n)}(0) = 0$ when n is even and $f^{(n)}(0) = 1$ when n is odd. So the Taylor series is

$$\frac{1}{1!}x^1 + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}.$$

We analyze the interval of convergence using the ratio test:

$$\begin{aligned}
\left|\frac{x^{2n+3}}{(2n+3)!}\frac{(2n+1)!}{x^{2n+1}}\right| &= \frac{x^2}{(2n+3)(2n+2)} \\
&\to 0
\end{aligned}$$

as $n \to \infty$, no matter what x is. Hence the interval of convergence is $(-\infty, \infty)$.