In physics, the angular coordinate θ of a complex number $re^{i\theta}$ is called its *phase*. Multiplying a complex number by a unit complex number $e^{i\phi}$ is called a *phase change*, because it changes the number's phase without changing its magnitude: $e^{i\phi}re^{i\theta} = re^{i(\phi+\theta)}$. Multiplying a complex vector by a unit complex number is called a *global phase change*, because it applies the same phase change to every component of the vector. For example, the transformation

$$\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} \mapsto e^{i\pi/7} \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} e^{i\pi/7}\psi_0 \\ e^{i\pi/7}\psi_1 \end{bmatrix}$$

is a global phase change, but the transformation

$$\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} \mapsto \begin{bmatrix} e^{i\pi/7}\psi_0 \\ e^{i\pi/5}\psi_1 \end{bmatrix}$$

is not a global phase change.

Exercise A. Let $|\psi\rangle \in \mathbb{C}^2$ and let $e^{i\phi}$ be any unit complex number. Prove that measurement of the state $e^{i\phi}|\psi\rangle$ has the same possible outcomes, with the same probabilities, as measurement of the state $|\psi\rangle$.

Exercise B. Let $|\psi\rangle$ and $e^{i\phi}$ be as above. Also let U be any 2×2 unitary matrix. Is it true that $Ue^{i\phi}|\psi\rangle = e^{i\phi}U|\psi\rangle$?

Exercise C. Using Exercises A and B, explain why $e^{i\phi}|\psi\rangle$ is indistinguishable from $|\psi\rangle$ in any 1-qbit quantum computation.

In class we defined the states of a qbit to be unit vectors in $\mathbb{C}^2$. That definition works well, as long as we remember that two states, that differ from each other by a global phase change, are the same state for all practical purposes, because they are physically indistinguishable.

If we wanted to be more formal, we could define the states of a qbit to be equivalence classes of unit vectors in $\mathbb{C}^2$, under the equivalence relation $|\psi\rangle \sim e^{i\phi}|\psi\rangle$ for all unit complex numbers $e^{i\phi}$. In practice, this definition amounts to pretty much the same thing as the previous definition.

Alternatively, we could define the states of a qbit to be equivalence classes of non-zero vectors in $\mathbb{C}^2$, under the equivalence relation that $|\psi\rangle \sim x|\psi\rangle$ for all non-zero complex numbers x. To readers who know projective geometry, this definition reveals the set of states to be the complex projective line $\mathbb{CP}^1$. Sometimes this definition is slightly more or less convenient than the others.

To keep the abstraction to a minimum, let's stick with our original definition. States are unit vectors. States that differ by only a global phase change are physically indistinguishable.

Exercise D. Let $|\psi\rangle \in \mathbb{C}^2$ be any 1-qbit state. Prove that there exist $\theta \in [0, 2\pi)$, $\alpha \in [0, 2\pi)$, and $\beta \in [0, \pi]$ such that

$$|\psi\rangle = e^{i\theta} \begin{bmatrix} \cos(\beta/2) \\ \sin(\beta/2)e^{i\alpha} \end{bmatrix}. \tag{1}$$

In Equation (1), the angle θ affects the state only through global phase change. So states of this form that have equal values for α and β are physically indistinguishable, even if they have differing values for θ.

Exercise E. Prove that all 1-qbit states of the form (1) with $\beta = 0$ are physically indistinguishable. Prove that all states of the form (1) with $\beta = \pi$ are physically indistinguishable. (It is also true, but you need not prove, that these are the only cases where differing values of α and β produce indistinguishable states.)

Exercise F. Explain why the set of physically distinguishable 1-qbit states forms a sphere.