A1. Well,

$$\begin{aligned} 1 &= \int_1^\infty cx^{-2}\,dx \\ &= c \lim_{b\to\infty} \left[-x^{-1}\right]_1^b \\ &= c \lim_{b\to\infty} \left(-\frac{1}{b}+1\right) \\ &= c. \end{aligned}$$

A2. As X varies from 1 to ∞, Y varies from 0 to $-\infty$. So the domain of f_Y is $(-\infty, 0]$.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(-\log X \leq y) \\ &= P(X \geq e^{-y}) \\ &= 1 - F_X(e^{-y}) \\ \Rightarrow f_Y(y) &= \frac{d}{dy}\left(1 - F_X(e^{-y})\right) \\ &= -f_X(e^{-y}) \cdot (-e^{-y}) \\ &= e^{2y} e^{-y} \\ &= e^y. \end{aligned}$$

[Notice that $F_Y = f_Y$. It is then easy to check that $\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$.]

B1. [This problem is a generalization of Exercise 4.38.] Using the definition of covariance and linearity of expectation, we compute

$$\begin{aligned} \text{Cov}(X+Y, X-Y) &= E[(X+Y)(X-Y)] - E[X+Y]E[X-Y] \\ &= E[X^2 - Y^2] - (E[X]+E[Y])(E[X]-E[Y]) \\ &= E[X^2] - E[Y^2] - E[X]^2 + E[Y]^2 \\ &= V[X] - V[Y]. \end{aligned}$$

So the covariance is 0 exactly when X and Y have equal variance.

B2. We must check that $P(X+Y=a, X-Y=b) = P(X+Y=a)P(X-Y=b)$ for all valid a, b. [We cannot carry out this check, because we know so little about X and Y.]

C. We model the time (in minutes) between arrivals using an exponentially distributed random variable X with parameter $\lambda = 20$. Then $E[X] = 1/\lambda = 1/20$ and $V[X] = 1/\lambda^2 = 1/400$.

The standard deviation is $SD[X] = \sqrt{V[X]} = 1/20$. As a rule of thumb, X will usually be in the interval $E[X] \pm 2SD[X]$. In this case, the interval extends into the negative numbers, indicating that it is not a great approximation. Keeping that caveat in mind, the time between arrivals will be around 3 seconds and usually between 0 seconds and 9 seconds.

D1. [Warning: Functions like this $cx(x+y)$ appear in predator-prey studies, but not as probability density (mass) functions, to my knowledge. That is, the problem set-up is not scientifically authentic.] Well,

$$\begin{aligned} P(X=x) &= \sum_{y=1}^{n} P(X=x, Y=y) \\ &= \sum_{y=1}^{n} cx(x+y) \\ &= cx^2 \sum_{y=1}^{n} 1 + cx \sum_{y=1}^{n} y \\ &= cx^2 n + cxn(n+1)/2 \end{aligned}$$

and

$$\begin{aligned} P(Y=y) &= \sum_{x=1}^{m} P(X=x, Y=y) \\ &= \sum_{x=1}^{m} cx(x+y) \\ &= c \sum_{x=1}^{m} x^2 + cy \sum_{x=1}^{m} x \\ &= cm(m+1)(2m+1)/6 + cym(m+1)/2. \end{aligned}$$

D2. By the definition of conditional probability,

$$P(X=x|Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{x(x+y)}{m(m+1)(2m+1)/6 + ym(m+1)/2}.$$