**A.A.** For each parameter, our Metropolis-Hastings algorithm proposes a new value by drawing from a normal distribution centered on the current value. Suppose that we're simulating $\sigma$, and the current value $\sigma_i$ is close to 0. Then there is a chance that the proposed $\sigma'$ will be zero or negative — a nonsensical result. The same danger applies to $\sigma^2$. But $\log \sigma$ can safely take on any real value, so it is not subject to this problem.

**A.B.** We do our usual transformation process. First,

$$F_L(\ell) = P(L \leq \ell) = P(\log \Sigma \leq \ell) = P(\Sigma \leq e^\ell) = F_\Sigma(e^\ell).$$

Then differentiation yields

$$f_L(\ell) = \frac{d}{d\ell} F_\Sigma(e^\ell) = f_\Sigma(e^\ell)e^\ell = e^{-\ell}e^\ell = 1.$$

**B.A.** The random variable $Y$ is supported on $(0, \infty)$, where it has density

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx = \int_0^y e^{-y}\, dx = ye^{-y}.$$

**B.B.** First, the conditional density is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{e^{-y}}{ye^{-y}} = y^{-1}.$$

The conditional expectation is then

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx = \int_0^y xy^{-1}\, dx = \left[\frac{1}{2}x^2y^{-1}\right]_{x=0}^{x=y} = y/2.$$

[Here's some intuition. The conditional density says that $X$ is uniform on $[0, y]$, which makes sense, because the joint density doesn't depend on $x$ at all. Then, because $X$ is uniform on $[0, y]$, its average value should be $y/2$.]

**B.C.** Because $E(X|Y = y) = y/2$, we immediately conclude that $E(X|Y) = Y/2$.

**C.A.** Using some algebra and linearity of expectation, we derive

$$m_Y(t) = E(e^{tY}) = E(e^{tX+tc}) = E(e^{ct}e^{tX}) = e^{ct}E(e^{tX}) = e^{ct}m_X(t).$$

**C.B.** Let $Z \sim \text{Norm}(0, 1)$, so $m_Z(t) = e^{t^2/2}$. Then, by problem C.A,

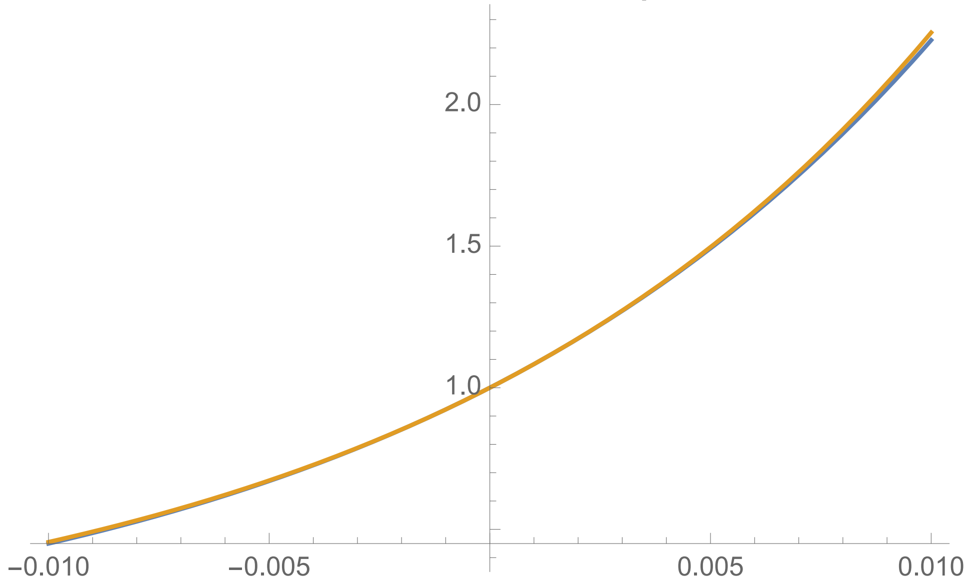
$$m_W(t) = m_{\mu+\sigma Z}(t) = e^{\mu t}m_{\sigma Z}(t) = e^{\mu t}m_Z(\sigma t) = e^{\mu t}e^{\sigma^2t^2/2} = e^{\mu t+\sigma^2t^2/2}.$$

**C.C.** If $X \sim \text{Bern}(p)$, then $m_X(t) = (1 - p) + e^tp$. [This fact can be cited from the course/book or re-derived quickly.] Then, because $T$ can be regarded as the sum of $n$ IID $\text{Bern}(p)$ random variables, $m_T(t) = (m_X(t))^n = ((1 - p) + e^tp)^n$.

**C.D.** We know that $T$ arises as a sum of $n$ IID random variables, $E(T) = np$, and $\text{Var}(T) = np(1-p)$. So the central limit theorem says that $T$ is approximately distributed as $\text{Norm}(np, np(1-p))$. And a random variable from that distribution has MGF

$$e^{npt+np(1-p)t^2/2},$$

by problem C.B. [It is not obvious that the answer to problem C.C and the answer to problem C.D are approximately equal. The graphs of the two functions are shown below, for $p = 0.8$ and $n = 100$. Indeed, they are nearly identical near $t = 0$.]




**D.** First,

$$\begin{aligned}
F_W(w) &= P(W \leq w) \\
&= P(XY \leq w) \\
&= P(Y \leq w/X) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{w/x} f_{X,Y}(x, y)\, dy\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{w/x} f_Y(y) f_X(x)\, dy\, dx \\
&= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{w/x} f_Y(y)\, dy \right) f_X(x)\, dx.
\end{aligned}$$

(The third equation holds because $X > 0$. The fifth equation holds because $X$ and $Y$ are independent.) Now we differentiate:

$$\begin{aligned} f_W(w) &= \frac{d}{dw} F_W(w) \\ &= \frac{d}{dw} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{w/x} f_Y(y)\, dy \right) f_X(x)\, dx \\ &= \int_{-\infty}^{\infty} \frac{d}{dw} \left( \int_{-\infty}^{w/x} f_Y(y)\, dy \right) f_X(x)\, dx \\ &= \int_{-\infty}^{\infty} f_Y(w/x) x^{-1} f_X(x)\, dx. \end{aligned}$$

(The final equation holds because of the fundamental theorem of calculus and the chain rule.)

**E.A.** FALSE. [The density $f_X$ is defined on the whole real number line, and $f_X(-1) = 0$.]

**E.B.** FALSE. [It should be $P(A|B) = P(A)$.]

**E.C.** FALSE. [It should be Pois($\lambda t$). A "number of arrivals" should be a discrete random variable.]

**E.D.** TRUE. [They are both uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$.]

**E.E.** FALSE. [$X^2$ is uniformly distributed on $\{1, 4, 9, 16, 25, 36\}$, while $XY$ is uniformly distributed on $\{6, 10, 12\}$.]

**F.A.** Let $A$, $B$, $C$ be the events that the air bag was made by Acme, Bagatronic, Carface, respectively. Let $D$ be the event that the air bag is defective. Then

$$\begin{aligned} P(D) &= P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) \\ &= 0.006 \cdot 0.50 + 0.002 \cdot 0.35 + 0.007 \cdot 0.15. \end{aligned}$$

[This number is about $0.00475 = 0.475\%$.]

**F.B.** Let $A$, $B$, $C$ be the events that the air bags were made by Acme, Bagatronic, Carface, respectively. Let $N$ be the event that none are defective. Then

$$P(N) = P(N|A)P(A) + P(N|B)P(B) + P(N|C)P(C).$$

Now assume that Acme air bags are independent of each other, in whether they are defective. [This assumption would be bad if, for example, your air bags came from a single batch and that batch had a systemic problem. But there is no way to complete the problem without such an assumption. The presence of such an assumption is indicative of the student's understanding.]

Then $P(N|A) = (1 - 0.006)^6$. Similarly, for the other manufacturers we assume independence and deduce that $P(N|B) = (1 - 0.002)^6$ and $P(N|C) = (1 - 0.007)^6$. Thus

$$P(N) = (1 - 0.006)^6 \cdot 0.50 + (1 - 0.002)^6 \cdot 0.35 + (1 - 0.007)^6 \cdot 0.15.$$

[This number is about $0.971898 \approx 97\%$.]

**F.C.** Establish the same notation $A$, $B$, $C$, $D$ as in problem F.A. Then

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)},$$

where $P(D)$ is exactly as in problem F.A. Thus we obtain

$$P(A|D) = \frac{0.006 \cdot 0.50}{0.006 \cdot 0.50 + 0.002 \cdot 0.35 + 0.007 \cdot 0.15}.$$

[This number is about $0.631579 \approx 63\%$.]

**G**. The customer's life span is $X \sim \text{Geom}(p)$, where "success" is dying in a given year and "failure" is not dying. The expected total payment is

$$E(100X) = 100E(X) = 100(1 - p)/p.$$