

Figure 1: A displacement vector representing a walk of 2 km to the northwest, starting from either a farm house or a nearby pond. (Can you perceive that one of the arrows is rasterized and the other is not? Did I do that for educational purposes? No, I'm just terrible at this drawing program.)

This document teaches you the basics of vectors: addition, scaling, and their geometric meaning. We work mainly in two dimensions, before shifting to arbitrary dimensions near the end. We also discuss the representation of vectors in  $\mathbb{C}$  as coordinate columns.

## 1 What is a vector?

There are various ways of thinking about vectors. In many sciences, and in this course, the most useful way is: A *vector* is a quantity possessing both magnitude and direction. To emphasize that they are different from numbers, vectors are often written with arrows over them, like  $\vec{v}$ , or in bold, like  $\mathbf{v}$ .

For example, imagine that you live in a farm house on an open plain. Starting from your house, you walk 2 km northwest. Your displacement is a vector, that we might denote  $\vec{v} = (2 \text{ km, northwest})$ . Its magnitude is 2 km, and its direction is northwest. See Fig. 1.

Let's emphasize that the vector does not describe your starting point or your ending point, but rather the difference between the two. If you start from a nearby pond and walk 2 km northwest from there, then your displacement is exactly the same vector  $\vec{v}$  as in the previous example. See Fig. 1 again.

On the other hand, if we agree that the farm house is the center of the world, then we can describe any point  $P$  in the plain using a vector  $\vec{v}$  — the displacement vector  $\vec{v}$  that takes you from the farm house to  $P$ . Conflating points with vectors like this is very common.

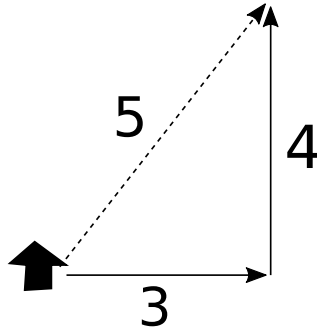


Figure 2: Walking 3 km east and then 4 km north produces a total displacement of 5 km in a particular direction which is approximately northeast.

## 2 Addition

Starting from your hypothetical farm house, walk 3 km to the east and then 4 km to the north. Your path consists of the vectors  $\vec{v} = (3 \text{ km, east})$  and  $\vec{w} = (4 \text{ km, north})$ . Your total displacement from the farm house is 5 km in a northeast-ish direction. We say that  $\vec{v} + \vec{w} = (5 \text{ km, northeast-ish})$ . See Fig. 2.

Alternatively, if you walk 4 km to the north and then 3 km to the east, you again end up 5 km northeast-ish from your starting point. That is,  $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ . This fact is true in general; addition of vectors is commutative.

## 3 Coordinates

It is extremely common to pick some “standard” directions, such as east and north, and express all vectors in terms of those directions. Returning to the preceding example, our  $\vec{v}$  could be written

$$\vec{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

meaning “3 km east and 0 km north”. The numbers 3 and 0 are called *coordinates* for the vector  $\vec{v}$ , and they are denoted  $v_1$  and  $v_2$ . Similarly,  $\vec{w}$  could be written

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix},$$

meaning “0 km east and 4 km north”, and  $\vec{v} + \vec{w}$  could be written

$$\vec{v} + \vec{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

meaning “3 km east and 4 km north”.

People like coordinates, because coordinates make computation with vectors simple and fast. But there is a pitfall in coordinates: You have to know what coordinate system you're using, and you have to apply it consistently. For example, there is nothing stopping you from picking north and west for your "standard" directions. In this coordinate system,

$$\vec{v} + \vec{w} = \begin{bmatrix} 4 \\ -3 \end{bmatrix},$$

meaning "4 km north and -3 km west". But surely it can't be true that

$$\vec{v} + \vec{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{v} + \vec{w} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

simultaneously, right? The vector  $\vec{v} + \vec{w}$  has a Platonic existence, independent of any coordinate system. If you compute coordinates for it, then those coordinates are meaningful only within their coordinate system. Columns of numbers defined with respect to differing coordinate systems are not comparable.

Notice that the eastern coordinate of  $\vec{v} + \vec{w}$  is the sum of the eastern coordinates of  $\vec{v}$  and  $\vec{w}$ , and likewise for the northern coordinates. That's no coincidence. In general, for any vectors  $\vec{v}$  and  $\vec{w}$ ,

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

Implicit in this statement is that the coordinates for  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} + \vec{w}$  must be defined relative to a single, consistent coordinate system.

## 4 Negation, zero, subtraction

Building on the preceding farm example, suppose that you walk the 5 km straight back to your house (Fig. 3). This displacement is the vector (5 km, southwest-ish). It is the *negation* of (5 km, northeast-ish). When you add these two vectors, you end up with a total displacement of (0 km, any direction), because you're back where you started your walk.

The vector (0 km, any direction) plays the role of zero in the world of vectors, and we denote it  $\vec{0}$ . In any coordinate system,  $\vec{0}$  has coordinates

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For any vector  $\vec{v}$ ,  $\vec{v} + \vec{0} = \vec{v}$ .

We denote the negation of  $\vec{v}$  as  $-\vec{v}$ . In any coordinate system, we can negate  $\vec{v}$  by simply negating its coordinates:

$$-\vec{v} = - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}.$$

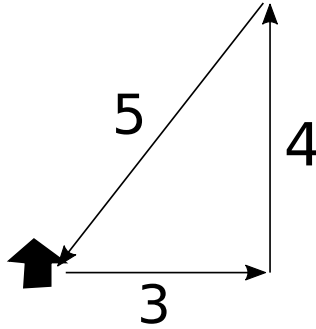


Figure 3: Ending the walk by going 5 km southwest-ish produces a new total displacement of  $\vec{0}$ , meaning that you're back where you started.

It's always true that  $\vec{v} + -\vec{v} = \vec{0}$ .

Vector *subtraction* is defined in terms of addition and negation:  $\vec{v} - \vec{w} = \vec{v} + -\vec{w}$ . In any coordinate system, subtraction amounts to subtracting the corresponding coordinates:

$$\vec{v} - \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \end{bmatrix}.$$

## 5 Scaling

Returning again to our farm example, suppose that on the next day you take a similar walk, but twice as long. That is, you walk 6 km east, then 8 km north, then 10 km southwest-ish. Each of these vectors corresponds to a vector from the previous day, but *scaled* by 2.

In general, scaling a vector  $\vec{v}$  by a number  $c$  produces a new vector, denoted  $c\vec{v}$ . This  $c\vec{v}$  has the same direction as  $\vec{v}$ , but its magnitude is  $c$  times as big. To clarify what that means, here are examples.

- If  $c > 1$ , then  $c\vec{v}$  is a stretched version of  $\vec{v}$ .
- If  $c = 1$ , then  $c\vec{v} = \vec{v}$ . Scaling by 1 has no effect.
- If  $0 < c < 1$ , then  $c\vec{v}$  is a shrunken version of  $\vec{v}$ .
- If  $c = 0$ , then  $c\vec{v} = \vec{0}$ .
- If  $-1 < c < 0$ , then  $c\vec{v}$  points in the opposite direction from  $\vec{v}$ , and is shrunken.
- If  $c = -1$ , then  $c\vec{v} = -\vec{v}$ , the negation.
- If  $c < -1$ , then  $c\vec{v}$  points in the opposite direction from  $\vec{v}$ , and is stretched.

Like other vector arithmetic, scaling is easy in coordinates:

$$c\vec{v} = c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}.$$

## 6 In arbitrary dimensions

Thus far, all of our examples have occurred in a 2-dimensional plane (or, metaphorically, a plain). However, vectors can be defined in any dimension. For example, if a bird flies 7 km east and 5 km south, and meanwhile gains 1 km of altitude, then its 3-dimensional displacement is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}$$

in the east-north-up coordinate system. In our course, we will primarily use 2-, 3-, and 4-dimensional vectors, but we will sometimes use higher dimensions. And by the way 1-dimensional vectors are just numbers.

Working in  $d$  dimensions and a fixed coordinate system, all of the arithmetic operations behave as you would expect. In particular,

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_d \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_d + w_d \end{bmatrix},$$

$$c\vec{v} = c \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_d \end{bmatrix}.$$

## 7 Algebraic properties

If you take a linear algebra course, you will see the following properties presented as the axioms for an abstract concept called a  $d$ -dimensional real “vector space”. In our course, you can instead view this list as a handy summary of the most important rules that vectors obey.

There exists a vector  $\vec{0}$  such that, for all numbers  $c$  and  $b$ , and for all vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ :

- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$  (additive associativity),
- $\vec{v} + \vec{0} = \vec{v} = \vec{0} + \vec{v}$  (additive identity),
- there exists  $-\vec{v}$  such that  $\vec{v} + -\vec{v} = \vec{0} = -\vec{v} + \vec{v}$  (additive inverses),
- $\vec{v} + \vec{w} = \vec{w} + \vec{v}$  (additive commutativity),

- $b(c\vec{v}) = (bc)\vec{v}$  (scalar associativity),
- $1\vec{v} = \vec{v}$  (scalar identity),
- $(b + c)\vec{v} = b\vec{v} + c\vec{v}$  (distributivity over scalar addition),
- $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$  (distributivity over vector addition).

Many other properties follow from the ones listed above, including  $0\vec{v} = \vec{0}$  and  $-1\vec{v} = -\vec{v}$ .

## 8 In C

Vectors are commonly represented in C as arrays of floating-point numbers. That is, the arrays store the coordinates of the vectors, in some coordinate system, of which the programmer must keep track. The length of the C array corresponds to the dimension of the vector. (In the world of vectors, the term “length” is another name for magnitude. Do not confuse the length of a vector with the length of its C array.)

You can imagine writing a C function to add two vectors. If your function were specialized to two dimensions, then it might look like this:

```
/* Adds the 2-dimensional vectors v and w. */
void vecAdd(double v[2], double w[2], double vPlusW[2]) {
    vPlusW[0] = v[0] + w[0];
    vPlusW[1] = v[1] + w[1];
}
```

If your function were intended to work in all dimensions, then you would have to pass the dimension as an argument, because C arrays do not know their lengths:

```
/* Adds the d-dimensional vectors v and w. */
void vecAdd(int d, double v[], double w[], double vPlusW[]) {
    ...
}
```

Notice that the coordinates of a  $d$ -dimensional vector  $\vec{v}$  are  $v_1, \dots, v_d$  in standard mathematical notation but  $v[0], \dots, v[d - 1]$  in C. So be careful when transcribing from one notation to the other. In this course, we will index vectors from 0, even in math notation, to reduce the rate of programming errors.