Prologue: These are some extra problems, that you might study as you prepare for our first exam. They are not intended to be indicative of what our exam will be like. For example, they might take much more than 70 minutes, and they might overly hard. (I don't know. I haven't done them in a while.) Anyway, give them a try, and don't get falsely alarmed that this is what the exam is.

1. Lines

Remember the problem about normal lines and circles? This problem is similar, but it's about tangent lines and lines.

Let $\vec{\gamma} : I \to \mathbb{R}^3$ be of unit speed. Prove that the image of $\vec{\gamma}$ is contained in a straight line if and only if there is a point $\vec{p} \in \mathbb{R}^3$ such that every tangent line of $\vec{\gamma}$ passes through $\vec{p}$.

2. Bézier Splines

Let $\vec{a}_0, \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^3$ be any four points. Define a parametrized curve $\vec{\alpha} : [0, 1] \to \mathbb{R}^3$ by the formula

$$\vec{\alpha}(t) = (1-t)^3\vec{a}_0 + 3t(1-t)^2\vec{a}_1 + 3t^2(1-t)\vec{a}_2 + t^3\vec{a}_3.$$

This is the cubic *Bézier curve* for the *control points* $\vec{a}_0, \vec{a}_1, \vec{a}_2, \vec{a}_3$. Suppose that $\vec{b}_0, \vec{b}_1, \vec{b}_2, \vec{b}_3$ is another sequence of control points, with Bézier curve $\vec{\beta}$. Under suitable conditions, $\vec{\alpha}$ and $\vec{\beta}$ can be joined end-to-end to construct a more complicated curve, which is then called a *spline*. Such splines are used heavily in computer graphics, such as for drawing the letters that you are reading right now. There is a similar notion of Bézier surfaces, that we may study later in the semester.

A. Compute the end points $\vec{\alpha}(0)$ and $\vec{\alpha}(1)$. Compute the tangent vectors $\vec{\alpha}'(0)$ and $\vec{\alpha}'(1)$. Using these results, draw a picture that illustrates how the points $\vec{a}_0, \vec{a}_1, \vec{a}_2, \vec{a}_3$ "control" $\vec{\alpha}$.

B. What conditions must the $\vec{a}_i$ and $\vec{b}_i$ meet, if $\vec{\alpha}(1)$ is to coincide with $\vec{\beta}(0)$? What conditions must they meet, if the two curves are to have the same tangent vector there?

3. Zero Curvature

Usually we prefer curves with nonvanishing curvature over those whose curvature vanishes somewhere. In this problem we explore that bias. We seek a curve $\vec{\alpha}(s)$, parametrized by arc length, with constant curvature $k(s) \equiv 0$ and constant torsion $\tau(s) \equiv 1$, that satisfies the Frenet equations with initial conditions $\vec{t}(0) = (0, 0, 1)$, $\vec{n}(0) = (1, 0, 0)$, and $\vec{b}(0) = (0, 1, 0)$.

A. Rewrite and simplify the Frenet equations using these assumptions.

B. Find one orthonormal solution $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ to the differential equations from Part A and the initial conditions.

C. What parametrized curve $\vec{\alpha}(s)$ arises from the solution to Part B? Give a formula. Also sketch the curve, indicating how the frame $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ moves along the curve.

D. How would the curve $\vec{\alpha}(s)$ have been different if I had specified $\tau(s) \equiv 0$ instead of $\tau(s) \equiv 1$? What is the role of torsion in this problem? (Probably at least a few sentences are required to answer these questions.)

4. Riparian Erosion

This problem uses parametrized curves to model erosion along a river. No knowledge of fluid dynamics or sediment transport is necessary. The model may or may not be scientifically accurate; compromises have been made to keep it tractable. Do not assume that the curves in this problem are parametrized by arc length.

A. As a prelude, prove that for any plane curve $\vec{\alpha}(t) = (x(t), y(t)) : \mathbb{R} \to \mathbb{R}^2$,

$$-\left|\vec{\alpha}'\right|^4 k\vec{n} = \left(y'(x'y'' - y'x''), -x'(x'y'' - y'x'')\right).$$

Viewed from a high altitude, a river flowing across a flat landscape looks a lot like a plane curve $\vec{\alpha}(t)$. That is, $\vec{\alpha}(t) = (x(t), y(t))$ is the position at time t of some molecule of water flowing along the river, and the other molecules pass through the same points at other times t. For example, a meandering river might be described by $\vec{\alpha}(t) = (t, \sin t)$, while a straight, fast river might be described by $\vec{\alpha}(t) = (10t, 0)$.

However, a river continually erodes its banks, so the course of the river changes gradually over time. So the river is really a one-parameter family of curves

$$\vec{\alpha}_s(t) = \vec{\alpha}(t, s) = (x(t, s), y(t, s)).$$

Although both t and s represent time, they have very different scales — seconds vs. years, perhaps — and we should view them as independent variables.

Here's how erosion works. At any bend in the river, the water tends to deposit sediment on the inside of the bend and remove sediment from the outside of the bend, for reasons that lie beyond the scope of this problem. Therefore the bend in the river moves; over time it comes to bend even further outward than it used to. This is why rivers develop meanders. In contrast, straight portions of the river do not change much as erosion progresses. One more thing: The rate of erosion increases dramatically as the speed of the water increases.

B. Based on the description of erosion just given, explain why $\vec{\alpha}(t, s) = (x(t, s), y(t, s))$ might satisfy the following differential equations. The prime $'$ now denotes $\partial/\partial t$, rather than d/dt.

$$\begin{aligned} \frac{\partial x}{\partial s} &= y'(x'y'' - y'x''), \\ \frac{\partial y}{\partial s} &= -x'(x'y'' - y'x''). \end{aligned}$$

Assuming that the initial course of the river is $\vec{\alpha}(t, 0) = (t, t^2)$, we will now solve the differential equations using power series. (If you have never done such stuff before, give it a try, and ask questions if necessary.) Suppose that $x(t,s)$ and $y(t,s)$ expand asymptotically as

$$\begin{aligned} x(t,s) &= \sum_{d=0}^{\infty} \sum_{i+j=d} a_{ij} t^i s^j &= a_{00} + a_{10}t + a_{01}s + a_{20}t^2 + a_{11}ts + a_{02}s^2 + \cdots, \\ y(t,s) &= \sum_{d=0}^{\infty} \sum_{i+j=d} b_{ij} t^i s^j &= b_{00} + b_{10}t + b_{01}s + b_{20}t^2 + b_{11}ts + b_{02}s^2 + \cdots. \end{aligned}$$

Finding x and y boils down to finding the constants a_{ij} and b_{ij}. To keep things tractable, forget about the $\cdots$, thereby truncating the series after the quadratic terms. In other words, let's just approximate x and y by their degree-2 Taylor polynomials.

C. What does the initial condition $\vec{\alpha}(t,0) = (t,t^2)$ tell you about the coefficients a_{ij} and b_{ij} in the Taylor polynomials? Use this information to rewrite the differential equations from Part B explicitly.

D. Solve for $y(t,s)$ and $x(t,s)$. Cubic terms may arise; ignoring them does not make your life easier, so do not ignore them. Your solutions will have one undetermined parameter, such as a_{11}.

E. Set the undetermined parameter equal to 1. (This is another initial condition, but not one worth explaining.) Give a simplified, exact formula for the curve of the river at time $s = 0.1$.

F. In a single graph, sketch the river at times $s = 0$ and $s = 0.1$, based on the solution to Part E. Your sketch should be precise enough that the x- and y-intercepts are approximately correct.