A. Differentiating $c^2 = \left|\vec{\gamma} - \vec{\delta}\right|^2$ produces

$$0 = (\vec{\gamma} - \vec{\delta})' \cdot (\vec{\gamma} - \vec{\delta}) + (\vec{\gamma} - \vec{\delta}) \cdot (\vec{\gamma} - \vec{\delta})' = 2(\vec{\gamma} - \vec{\delta}) \cdot (\vec{\gamma}' - \vec{\delta}'),$$

which implies that

$$\vec{\gamma}' \cdot (\vec{\gamma} - \vec{\delta}) = \vec{\delta}' \cdot (\vec{\gamma} - \vec{\delta}).$$

Let θ, ϕ be the two angles in question, so that the equation above becomes

$$\left|\vec{\gamma}'\right| c \cos\theta = \left|\vec{\delta}'\right| c \cos\phi.$$

Because $|\vec{\gamma}'| = \left|\vec{\delta}'\right| = 1$, we get $\cos\theta = \cos\phi$. Because θ and ϕ are in $[0, \pi]$, we conclude that $\theta = \phi$.

B.A. The curve $\vec{\delta}$ is closed because all derivatives of $\vec{\gamma}$ match at $\pm 2\pi$. There is no reason for it to be simple or unit-speed. [Draw examples.]

B.B. There is no reason for $\vec{\beta}$ to be simple, closed, or unit-speed. For example, if $\vec{\gamma}$ is entirely in the right half of the plane, then $\vec{\beta}$ will not close. For another example, imagine that $\vec{\beta}$ is the figure eight parametrized at unit speed. Then $\vec{\gamma}$ is a non-simple closed curve in $\mathbb{S}^1$. Perturb $\vec{\gamma}$ slightly, so that it is a simple closed curve in $\mathbb{R}^2$. Parametrize the perturbed $\vec{\gamma}$ by arc length. Then $\vec{\gamma}$ fits the assumptions of the problem, but $\vec{\beta}$ is still a figure eight and not simple. [Draw examples.]

B.C. Because $\vec{\gamma}$ is a simple, closed, unit-speed curve, Hopf's Umlaufsatz says that the rotation index of $\vec{\gamma}$, which equals the degree of $\vec{\delta}$ as a map into $\mathbb{S}^1$, is ± 1. On the other hand, the degree of $\vec{\epsilon}$ as a map into $\mathbb{S}^1$ is 2. By Lemma 2.4, there is some $t \in [-2\pi, 2\pi]$ such that $\vec{\delta}(t) = -\vec{\epsilon}(t)$. The result follows.

C.A. Because $\kappa > 0$ and $|\vec{v}| = 1$, the simplified Frenet equations

$$\begin{aligned} \vec{t}' &= \kappa\vec{n}, \\ \vec{n}' &= -\kappa\vec{t} + \tau\vec{b}, \\ \vec{b}' &= -\tau\vec{n} \end{aligned}$$

hold. Then, using the assumption that κ and τ are constant, we differentiate to obtain

$$\vec{t}'' = \kappa\vec{n}' = -\kappa^2\vec{t} + \kappa\tau\vec{b}$$

and

$$\vec{t}''' = -\kappa^2\vec{t}' + \kappa\tau\vec{b}' = -\kappa^3\vec{n} - \kappa\tau^2\vec{n} = -(\kappa^2 + \tau^2)\kappa\vec{n} = -c^2\vec{t}',$$

where $c = \sqrt{\kappa^2 + \tau^2}$.

C.B. Applying the hint to each component of $\vec{t}'$, we have

$$\vec{t}' = \begin{bmatrix} a_1 \cos(ct) + b_1 \sin(ct) \\ a_2 \cos(ct) + b_2 \sin(ct) \\ a_3 \cos(ct) + b_3 \sin(ct) \end{bmatrix}$$

for some constants a_i, b_i. And then

$$\vec{t} = \begin{bmatrix} \frac{a_1}{c} \sin(ct) - \frac{b_1}{c} \cos(ct) + k_1 \\ \frac{a_2}{c} \sin(ct) - \frac{b_2}{c} \cos(ct) + k_2 \\ \frac{a_3}{c} \sin(ct) - \frac{b_3}{c} \cos(ct) + k_3 \end{bmatrix}$$

for some constants k_i. Finally,

$$\vec{\gamma} = \begin{bmatrix} -\frac{a_1}{c^2} \cos(ct) - \frac{b_1}{c^2} \sin(ct) + k_1 t + \ell_1 \\ -\frac{a_2}{c^2} \cos(ct) - \frac{b_2}{c^2} \sin(ct) + k_2 t + \ell_2 \\ -\frac{a_3}{c^2} \cos(ct) - \frac{b_3}{c^2} \sin(ct) + k_3 t + \ell_3 \end{bmatrix}$$

for some constants ℓ_i. The requirement that $\vec{t} \cdot \vec{t} = 1$ places some restrictions on the values of the unknown constants, but those restrictions are quite complicated, so we do not pursue them.