This exam is intended to take no more than 70 minutes. I am allowing 300 minutes (5 hours) so that you can relax a bit.

The exam is open-book and open-note — meaning:

- You may use all of this course's materials: the Tapp textbook, your class notes, your old homework, and the course web/Moodle site. You may also view official Mathematica reference/tutorial material, as long as it has nothing to do with differential geometry.

- You may not share any materials with any classmates while you are taking the exam. You may not consult any other books, papers, Internet sites, etc. You may not discuss the exam in any way — spoken, written, etc. — with anyone but me, until everyone has handed in the exam.

Your solutions should be thorough, self-explanatory, neat, concise, and polished. Always show enough work and justification so that a typical classmate could understand your solutions. Feel free to cite material (definitions, theorems, examples, etc.) from class, the assigned textbooks readings, and the assigned homework problems. On the other hand, you may not cite results that we have not studied.

If you cannot solve a problem, then write a brief summary of the approaches you've tried. Partial credit is often awarded. Correct answers without supporting work rarely earn full credit.

I will try to check my e-mail frequently during the exam period. Feel free to ask clarifying questions. If you cannot obtain clarification on a problem, then explain your interpretation of the problem in your solution. Never interpret a problem in a way that renders it trivial. Check your e-mail occasionally, in case I send out a correction.

Good luck. :)

Your phone rings, and you don't recognize the number, but you answer anyway. The caller is your great-great-great-great-great-great-great-great-great-great-great-great-great-grandchild, Babatope, in the year 2520, placing a back-in-time phone call. Babatope lives on a giant space station orbiting Venus. Because of how it generates artificial gravity for its inhabitants, the space station is shaped like a doughnut or bagel. In other words, its surface is a torus. Babatope's job is to promote tourism, and they want to make an accurate map of the space station's surface. They remembered (via an artificial intelligence that tracks precious family memories) that you knew something about accurate maps in 2020, and that's why they have just called you.

A. What advice (related to Math 344) can you give to Babatope about making an accurate map of the space station's surface?

For the second problem, recall that a quantity computed in local coordinates on U is *intrinsic* if it is determined by the metric $\mathbf{G}$.

B.A. Is the first fundamental form intrinsic?

B.B. Is the second fundamental form intrinsic?

For the third problem, let $\vec{f} : U \to S \subseteq \mathbb{R}^3$ be a diffeomorphic parametrized surface. Let $\vec{\phi} : U \to U$ be a diffeomorphism. (Notice that $\vec{\phi}$ is itself a diffeomorphic parametrized surface in $\mathbb{R}^2$.) Define $\tilde{\vec{f}} = \vec{f} \circ \vec{\phi} : U \to S$. That is, $\vec{f}$ and $\tilde{\vec{f}}$ are two diffeomorphic parametrizations of the same $S \subseteq \mathbb{R}^3$. Let Γ_{ij}^k be the Christoffel symbols for $\vec{f}$ and $\tilde{\Gamma}_{ij}^k$ the Christoffel symbols for $\tilde{\vec{f}}$.

C.A. By expanding

$$\frac{\partial}{\partial x_1} \frac{\partial \tilde{\vec{f}}}{\partial x_1}$$

in two ways, find two linear equations relating the Γ_{ij}^k to the $\tilde{\Gamma}_{ij}^k$.

C.B. How many linear equations do you expect to need, to solve for the $\tilde{\Gamma}_{ij}^k$ in terms of the Γ_{ij}^k? How would you obtain those equations? (Your answer should be a brief, specific plan — not a gigantic calculation.)

C.C. Explain why the following argument is invalid: "Whether we're talking about $\vec{f}$ or $\tilde{\vec{f}}$, the surface S is the same. The Riemannian metric on S is simply the restriction of the dot product in $\mathbb{R}^3$ to the tangent planes. Therefore the two metrics must be identical: $\mathbf{G}(\vec{p}) = \tilde{\mathbf{G}}(\vec{p})$ for all $\vec{p} \in U$. Then, because the Christoffel symbols are intrinsic, we must have $\Gamma_{ij}^k(\vec{p}) = \tilde{\Gamma}_{ij}^k(\vec{p})$ too."