This exam is intended to take no more than 70 minutes. I am allowing 300 minutes (5 hours) so that you can relax a bit.

The exam is open-book and open-note — meaning:

- You may use all of this course's materials: the Tapp textbook, your class notes, your old homework, and the course web/Moodle site. You may also view official Mathematica reference/tutorial material, as long as it has nothing to do with differential geometry.
- You may not share any materials with any classmates while you are taking the exam. You may not consult any other books, papers, Internet sites, etc. You may not discuss the exam in any way — spoken, written, etc. — with anyone but me, until everyone has handed in the exam.

Your solutions should be thorough, self-explanatory, neat, concise, and polished. Always show enough work and justification so that a typical classmate could understand your solutions. Feel free to cite material (definitions, theorems, examples, etc.) from class, the assigned textbooks readings, and the assigned homework problems. On the other hand, you may not cite results that we have not studied.

If you cannot solve a problem, then write a brief summary of the approaches you've tried. Partial credit is often awarded. Correct answers without supporting work rarely earn full credit.

I will try to check my e-mail frequently during the exam period. Feel free to ask clarifying questions. If you cannot obtain clarification on a problem, then explain your interpretation of the problem in your solution. Never interpret a problem in a way that renders it trivial. Check your e-mail occasionally, in case I send out a correction.

Good luck. :)

For the first problem, let $\vec{\sigma} : U \to S \subseteq \mathbb{R}^3$ be a surface chart. Let $\vec{\delta} : (-\epsilon, \epsilon) \to \vec{\sigma}(U) \subseteq S$ be a regular curve in the chart. Let $\vec{\gamma} = \vec{\sigma}^{-1} \circ \vec{\delta}$ be the corresponding regular curve in U.

A. Find a set of differential equations involving $\vec{\gamma}$ and its derivatives, such that $\vec{\gamma}$ satisfies the differential equations if and only if $\vec{\delta}$ is a geodesic in S. (Hint: Homework 18.)

For the next problem, let $\vec{\sigma} : U \to S$ and $\tilde{\vec{\sigma}} : \tilde{U} \to S$ be two charts on a regular surface $S \subseteq \mathbb{R}^m$. Suppose that they overlap, with $\vec{\sigma}(\vec{p}) = \tilde{\vec{\sigma}}(\tilde{\vec{p}}) = \vec{q} \in S$. Let $\vec{v}, \tilde{\vec{v}} \in \mathbb{R}^2$ be such that

$$d\vec{\sigma}_{\vec{p}}(\vec{v}) = d\tilde{\vec{\sigma}}_{\tilde{\vec{p}}}(\tilde{\vec{v}}) \in T_{\vec{q}}S.$$

B.A. Find an expression for $\tilde{\vec{v}}$ in terms of $\vec{v}$ (and the Jacobians, the metrics, etc.).
B.B. Find an expression for $\tilde{\mathbf{G}}(\tilde{\vec{p}})$ in terms of $\mathbf{G}(\vec{p})$ (and the Jacobians, etc.).
B.C. Do your answers still work if S is an n-dimensional manifold?

For the next problem, define a *regular surface with boundary* to be a set $S \subseteq \mathbb{R}^m$ such that each point $\vec{q} \in S$ has a neighborhood that is diffeomorphic to an open set in the half plane

$$H = \{(x_1, x_2) : x_2 \geq 0\} \subseteq \mathbb{R}^2.$$

The *boundary* is the set of points $\vec{q} \in S$ that do not possess a neighborhood that is diffeomorphic to an open set in $\mathbb{R}^2$. For example, the closed cylinder

$$C = \{(y_1, y_2, y_3) : y_1^2 + y_2^2 = 1, -1 \leq y_3 \leq 1\} \subseteq \mathbb{R}^3$$

is a regular surface with boundary. Its boundary consists of two circles at $y_3 = \pm 1$. It is compact. For another example, the half-closed cylinder

$$\{(y_1, y_2, y_3) : y_1^2 + y_2^2 = 1, -1 < y_3 \leq 1\} \subseteq \mathbb{R}^3$$

is a regular surface with boundary. Its boundary consists of one circle at $y_3 = 1$, and it is not compact. For a final example, any regular surface is a regular surface with boundary, where the boundary is empty.

Let $S \subseteq \mathbb{R}^3$ be the closed Möbius strip — like the closed cylinder with a twist. It is a compact regular surface with boundary. Its boundary is a simple closed curve. This S is not orientable.

C.A. Does the conclusion of the global Gauss-Bonnet theorem hold for the closed cylinder C?
C.B. If R is a polygonal region in S, then does the conclusion of the local Gauss-Bonnet theorem hold for R?
C.C. Does the conclusion of the global Gauss-Bonnet theorem hold for S?