A. Notice that $\vec{w} = \vec{\gamma}'$ is a vector field along $\vec{\gamma}$, which corresponds to the vector field $\vec{v} = \vec{\delta}'$ along $\vec{\delta}$. And $\vec{\delta}$ is a geodesic if and only if the covariant derivative of $\vec{v}$ is $\vec{0}$. Homework 18 Problem C.C tells us that the covariant derivative of $\vec{v}$ is

$$\vec{v}^{\not\prime}(t) \quad = \quad a(t)\frac{\partial \vec{f}}{\partial x_1}(\vec{\gamma}(t)) + b(t)\frac{\partial \vec{f}}{\partial x_2}(\vec{\gamma}(t))$$

where

$$\begin{aligned} a &= \gamma_1'' + \gamma_1'\gamma_1'\Gamma_{11}^1 + \left(\gamma_1'\gamma_2' + \gamma_2'\gamma_1'\right)\Gamma_{12}^1 + \gamma_2'\gamma_2'\Gamma_{22}^1, \\ b &= \gamma_2'' + \gamma_1'\gamma_1'\Gamma_{11}^2 + \left(\gamma_1'\gamma_2' + \gamma_2'\gamma_1'\right)\Gamma_{12}^2 + \gamma_2'\gamma_2'\Gamma_{22}^2. \end{aligned}$$

So $\vec{\delta}$ is a geodesic if and only if $\vec{\gamma}$ satisfies

$$\begin{aligned} 0 &= \gamma_1'' + \left(\gamma_1'\right)^2\Gamma_{11}^1 + 2\gamma_1'\gamma_2'\Gamma_{12}^1 + \left(\gamma_2'\right)^2\Gamma_{22}^1, \\ 0 &= \gamma_2'' + \left(\gamma_1'\right)^2\Gamma_{11}^2 + 2\gamma_1'\gamma_2'\Gamma_{12}^2 + \left(\gamma_2'\right)^2\Gamma_{22}^2. \end{aligned}$$

B.A. The premise of the problem is that

$$\frac{\partial\vec{\sigma}}{\partial\vec{x}} \cdot \vec{v} = \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}} \cdot \tilde{\vec{v}}.$$

Unfortunately, we can't invert these Jacobians, because they are $m \times 2$. But we can transpose and multiply:

$$\frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}}^\top \cdot \frac{\partial\vec{\sigma}}{\partial\vec{x}} \cdot \vec{v} = \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}}^\top \cdot \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}} \cdot \tilde{\vec{v}} = \tilde{\mathbf{G}} \cdot \tilde{\vec{v}}.$$

Then $\tilde{\mathbf{G}}$ is invertible because it is positive-definite. So we obtain

$$\tilde{\vec{v}} = \tilde{\mathbf{G}}^{-1} \cdot \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}}^\top \cdot \frac{\partial\vec{\sigma}}{\partial\vec{x}} \cdot \vec{v}.$$

B.B. For all $\vec{v}, \vec{w} \in \mathbb{R}^2$ with corresponding $\tilde{\vec{v}}, \tilde{\vec{w}} \in \mathbb{R}^2$,

$$\begin{aligned} \vec{v}^\top \cdot \mathbf{G} \cdot \vec{w} &= \tilde{\vec{v}}^\top \cdot \tilde{\mathbf{G}} \cdot \tilde{\vec{w}} \\ &= \left(\vec{v}^\top \cdot \frac{\partial\vec{\sigma}}{\partial\vec{x}}^\top \cdot \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}} \cdot \left(\tilde{\mathbf{G}}^{-1}\right)^\top\right) \cdot \tilde{\mathbf{G}} \cdot \left(\tilde{\mathbf{G}}^{-1} \cdot \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}}^\top \cdot \frac{\partial\vec{\sigma}}{\partial\vec{x}} \cdot \vec{w}\right) \\ &= \vec{v}^\top \cdot \frac{\partial\vec{\sigma}}{\partial\vec{x}}^\top \cdot \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}} \cdot \tilde{\mathbf{G}}^{-1} \cdot \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}}^\top \cdot \frac{\partial\vec{\sigma}}{\partial\vec{x}} \cdot \vec{w}. \end{aligned}$$

Because this equation holds for all $\vec{v}, \vec{w}$, the 2×2 matrices must be equal:

$$\mathbf{G} = \frac{\partial\vec{\sigma}}{\partial\vec{x}}^\top \cdot \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}} \cdot \tilde{\mathbf{G}}^{-1} \cdot \frac{\partial\tilde{\vec{\sigma}}}{\partial\tilde{\vec{x}}}^\top \cdot \frac{\partial\vec{\sigma}}{\partial\vec{x}}.$$

Because the relationship between $\vec{\sigma}$ and $\tilde{\vec{\sigma}}$ is symmetric, we can switch the tilded and non-tilded factors:

$$\tilde{\mathbf{G}} = \frac{\partial \tilde{\vec{\sigma}}}{\partial \tilde{\vec{x}}}^\top \cdot \frac{\partial \vec{\sigma}}{\partial \vec{x}} \cdot \mathbf{G}^{-1} \cdot \frac{\partial \vec{\sigma}}{\partial \vec{x}}^\top \cdot \frac{\partial \tilde{\vec{\sigma}}}{\partial \tilde{\vec{x}}}.$$

B.C. Yes, all of the work above immediately generalizes. The Jacobians are $m \times n$, $\tilde{\mathbf{G}}$ is positive-definite, and so forth. We end up with the same expressions for $\tilde{\vec{v}}$ and $\tilde{\mathbf{G}}$.

C.A. [Several students misinterpreted Problem C.A (and C.B and C.C). The problem is not asking whether the assumptions of the global Gauss-Bonnet theorem are met. They are not met, because the closed cylinder is not a regular surface at all. The problem is asking whether the integral of K equals $2\pi\chi$ anyway. A basic answer shows the computation. A really strong answer discusses how the proof works even though the assumptions are violated.]

We can triangulate C using 6 vertices, 12 edges, and 6 triangles. So the Euler characteristic is 0. Meanwhile, the Gaussian curvature of the cylinder is 0. So, yes, the theorem holds. To understand more deeply, let's examine how the proof of the theorem behaves for this C.

One issue is that the 6 boundary edges each belong to just one triangle, so we don't get cancellation of the integrals of κ_g. However, these boundary edges are geodesics, so $\kappa_g = 0$ along them, so those integrals vanish anyway.

Another issue is that the signed angles don't add up in the same way for a compact surface. At each vertex in the triangulation above, 3 triangles meet, and their interior angles sum to π, and it follows that the sum of the signed angles is 2π. So when we sum the signed angles over all vertices, we get 12π, which equals $2\pi(E - V)$, just as in the case of a compact surface, even though the logic along the way was different.

C.B. Yes, the local Gauss-Bonnet theorem holds, because the polygonal region R is contained in one chart, and any one chart is orientable — even if the surface as a whole is not orientable.

C.C. [This problem is intentionally imprecise. Your thought process is more important to me than the answer.]

We don't have a specific parametrization of the Möbius strip, so we don't know what its Gaussian curvature K should be. We could use the parametrization in Section 3.5 of our textbook:

$$\vec{\sigma}(\vec{x}) = \begin{bmatrix} (\cos x_1)(2 + x_2 \sin(x_1/2)) \\ (\sin x_1)(2 + x_2 \sin(x_1/2)) \\ x_2 \cos(x_1/2) \end{bmatrix}.$$

The calculations from here get pretty complicated, even for Mathematica. But I think that I've computed the determinant of the differential of the Gauss map as

$$K = \frac{-16}{(16 + 3x_2^2 - 2x_2^2 \cos(x_1) + 16x_2 \sin(x_1/2))^2} < 0.$$

For intuition, if I make a Möbius strip with its twist concentrated in a small part — so that most of the Möbius strip looks like a cylinder — then the cylinder part has $K = 0$ but the twisty part looks like a saddle and so should have $K < 0$. Anyway, it seems that the integral of K should be negative.

Meanwhile, a triangulation similar to the one above shows that the Euler characteristic is $\chi = 0$. So it seems that the Gauss-Bonnet theorem does not hold. [I invite you to consider where the proof breaks down, as I did for Problem C.A above. Also, some students came up with parametrizations that, they claimed, had $K > 0$. And some students argued, using the concept of rules surfaces, that $K = 0$.]