The Frenet equations capture deep knowledge of how curves in $\mathbb{R}^3$ behave. In particular, they lead to the following result, which says approximately that curvature and torsion uniquely determine a curve.

Fundamental Theorem of Curves in $\mathbb{R}^3$: Let I be an interval containing 0. Let $\tilde{\kappa} : I \to (0, \infty)$ and $\tilde{\tau} : I \to \mathbb{R}$ be smooth. Let $\tilde{\boldsymbol{\gamma}}(0) \in \mathbb{R}^3$ and let $\{\tilde{\mathbf{t}}(0), \tilde{\mathbf{n}}(0), \tilde{\mathbf{b}}(0)\}$ be a right-handed orthonormal basis of $\mathbb{R}^3$. Then there exists an $\epsilon > 0$ and a unique regular curve $\boldsymbol{\gamma} : (-\epsilon, \epsilon) \to \mathbb{R}^n$ such that

1. $\boldsymbol{\gamma}$ is parametrized by arc length measured from $\boldsymbol{\gamma}(0)$,
2. the initial conditions $\boldsymbol{\gamma}(0) = \tilde{\boldsymbol{\gamma}}(0)$, $\mathbf{t}(0) = \tilde{\mathbf{t}}(0)$, $\mathbf{n}(0) = \tilde{\mathbf{n}}(0)$, $\mathbf{b}(0) = \tilde{\mathbf{b}}(0)$ are satisfied, and
3. $\kappa(s) = \tilde{\kappa}(s)$ and $\tau(s) = \tilde{\tau}(s)$ for all $s \in (-\epsilon, \epsilon)$.

Proof: First, consider the first-order ordinary differential equations

$$\begin{aligned}
\tilde{\mathbf{t}}' &= \tilde{\kappa}\tilde{\mathbf{n}}, \\
\tilde{\mathbf{n}}' &= -\tilde{\kappa}\tilde{\mathbf{t}} + \tilde{\tau}\tilde{\mathbf{b}}, \\
\tilde{\mathbf{b}}' &= -\tilde{\tau}\tilde{\mathbf{n}}.
\end{aligned}$$

with respect to the independent variable s, subject to prescribed initial values $\tilde{\mathbf{t}}(0)$, $\tilde{\mathbf{n}}(0)$, $\tilde{\mathbf{b}}(0)$ at $s = 0$. The theory of existence and uniqueness of solutions to ordinary differential equations (Appendix A.3) says that there exists a unique smooth solution $\{\tilde{\mathbf{t}}(s), \tilde{\mathbf{n}}(s), \tilde{\mathbf{b}}(s)\}$ on a small interval $(-\epsilon, \epsilon)$.

Now consider the s-dependent symmetric matrix

$$\tilde{\mathbf{P}} = \begin{bmatrix} \tilde{\mathbf{t}} \cdot \tilde{\mathbf{t}} & \tilde{\mathbf{t}} \cdot \tilde{\mathbf{n}} & \tilde{\mathbf{t}} \cdot \tilde{\mathbf{b}} \\ \tilde{\mathbf{n}} \cdot \tilde{\mathbf{t}} & \tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} & \tilde{\mathbf{n}} \cdot \tilde{\mathbf{b}} \\ \tilde{\mathbf{b}} \cdot \tilde{\mathbf{t}} & \tilde{\mathbf{b}} \cdot \tilde{\mathbf{n}} & \tilde{\mathbf{b}} \cdot \tilde{\mathbf{b}} \end{bmatrix}.$$

The derivative with respect to s is

$$\begin{aligned}
\tilde{\mathbf{P}}' &= \begin{bmatrix} \tilde{\mathbf{t}}' \cdot \tilde{\mathbf{t}} + \tilde{\mathbf{t}} \cdot \tilde{\mathbf{t}}' & \tilde{\mathbf{t}}' \cdot \tilde{\mathbf{n}} + \tilde{\mathbf{t}} \cdot \tilde{\mathbf{n}}' & \tilde{\mathbf{t}}' \cdot \tilde{\mathbf{b}} + \tilde{\mathbf{t}} \cdot \tilde{\mathbf{b}}' \\ \tilde{\mathbf{n}}' \cdot \tilde{\mathbf{t}} + \tilde{\mathbf{n}} \cdot \tilde{\mathbf{t}}' & \tilde{\mathbf{n}}' \cdot \tilde{\mathbf{n}} + \tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}}' & \tilde{\mathbf{n}}' \cdot \tilde{\mathbf{b}} + \tilde{\mathbf{n}} \cdot \tilde{\mathbf{b}}' \\ \tilde{\mathbf{b}}' \cdot \tilde{\mathbf{t}} + \tilde{\mathbf{b}} \cdot \tilde{\mathbf{t}}' & \tilde{\mathbf{b}}' \cdot \tilde{\mathbf{n}} + \tilde{\mathbf{b}} \cdot \tilde{\mathbf{n}}' & \tilde{\mathbf{b}}' \cdot \tilde{\mathbf{b}} + \tilde{\mathbf{b}} \cdot \tilde{\mathbf{b}}' \end{bmatrix} \\
&= \begin{bmatrix} 2\tilde{\kappa}\tilde{\mathbf{t}} \cdot \tilde{\mathbf{n}} & -\tilde{\kappa}\tilde{\mathbf{t}} \cdot \tilde{\mathbf{t}} + \tilde{\tau}\tilde{\mathbf{b}} \cdot \tilde{\mathbf{t}} + \tilde{\kappa}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} & -\tilde{\tau}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{t}} + \tilde{\kappa}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{b}} \\ -\tilde{\kappa}\tilde{\mathbf{t}} \cdot \tilde{\mathbf{t}} + \tilde{\tau}\tilde{\mathbf{b}} \cdot \tilde{\mathbf{t}} + \tilde{\kappa}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} & -2\tilde{\kappa}\tilde{\mathbf{t}} \cdot \tilde{\mathbf{n}} + 2\tilde{\tau}\tilde{\mathbf{b}} \cdot \tilde{\mathbf{n}} & -\tilde{\tau}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} + \tilde{\kappa}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{b}} \\ -\tilde{\tau}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{t}} + \tilde{\kappa}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{b}} & -\tilde{\tau}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} + \tilde{\kappa}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{b}} & -2\tilde{\tau}\tilde{\mathbf{n}} \cdot \tilde{\mathbf{b}} \end{bmatrix} \\
&= \begin{bmatrix} 2\tilde{\kappa}\tilde{P}_{12} & -\tilde{\kappa}\tilde{P}_{11} + \tilde{\tau}\tilde{P}_{13} + \tilde{\kappa}\tilde{P}_{22} & -\tilde{\tau}\tilde{P}_{12} + \tilde{\kappa}\tilde{P}_{13} \\ -\tilde{\kappa}\tilde{P}_{11} + \tilde{\tau}\tilde{P}_{13} + \tilde{\kappa}\tilde{P}_{22} & -2\tilde{\kappa}\tilde{P}_{12} + 2\tilde{\tau}\tilde{P}_{23} & -\tilde{\kappa}\tilde{P}_{13} + \tilde{\tau}\tilde{P}_{33} - \tilde{\tau}\tilde{P}_{22} \\ -\tilde{\tau}\tilde{P}_{13} + \tilde{\kappa}\tilde{P}_{23} & -\tilde{\kappa}\tilde{P}_{13} + \tilde{\tau}\tilde{P}_{33} - \tilde{\tau}\tilde{P}_{22} & -2\tilde{\tau}\tilde{P}_{23} \end{bmatrix}.
\end{aligned}$$

With the symmetry conditions $\tilde{P}_{12} = \tilde{P}_{21}$, $\tilde{P}_{23} = \tilde{P}_{32}$, and $\tilde{P}_{13} = \tilde{P}_{31}$, as well as the initial conditions $\tilde{\mathbf{P}}(0) = \mathbf{I}$, the theory of ordinary differential equations again says that there exists a unique solution for $\tilde{\mathbf{P}}$ on an interval $(-\epsilon, \epsilon)$ (possibly smaller than the last interval). But, if you plug $\tilde{\mathbf{P}} \equiv \mathbf{I}$ into this expression for $\tilde{\mathbf{P}}'$, then you get $\tilde{\mathbf{P}}' = \mathbf{0}$, which is consistent with $\tilde{\mathbf{P}} = \mathbf{I}$. Hence the unique solution is simply $\tilde{\mathbf{P}} = \mathbf{I}$. That is, $\{\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}\}$ is an orthonormal basis for all $s \in (-\epsilon, \epsilon)$. Let $\tilde{\mathbf{F}}$ be the matrix with columns $\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}$. Because those vectors are orthonormal, $\tilde{\mathbf{F}}$ is orthogonal and $\det \tilde{\mathbf{F}} = \pm 1$. The determinant is a continuous function of s with value 1 at $s = 0$, so $\det \tilde{\mathbf{F}} = 1$. That is, the orthonormal basis is right-handed.

Finally let $\boldsymbol{\gamma}(s) = \boldsymbol{\gamma}(0) + \int_0^s \tilde{\mathbf{t}}(u)\, du$. By the fundamental theorem of calculus, $\boldsymbol{\gamma}'(s) = \tilde{\mathbf{t}}(s)$, which has norm 1 by the preceding paragraph. Hence $\boldsymbol{\gamma}$ is arc-length-parametrized and regular, and $\boldsymbol{\gamma}' = \mathbf{v} = \mathbf{t} = \tilde{\mathbf{t}}$. From the Frenet equations,

$$\kappa \mathbf{n} = \mathbf{t}' = \tilde{\mathbf{t}}' = \tilde{\kappa} \tilde{\mathbf{n}}.$$

Because $\tilde{\kappa} > 0$, it follows that $\kappa = \tilde{\kappa}$ and $\mathbf{n} = \tilde{\mathbf{n}}$. Then

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \tilde{\mathbf{t}} \times \tilde{\mathbf{n}} = \tilde{\mathbf{b}}$$

and

$$-\tau \mathbf{n} = \mathbf{b}' = \tilde{\mathbf{b}}' = -\tilde{\tau} \tilde{\mathbf{n}} = -\tilde{\tau} \mathbf{n},$$

which implies that $\tau = \tilde{\tau}$. So $\boldsymbol{\gamma}$ solves the problem. Moreover, $\tilde{\mathbf{t}}$ is unique, and $\boldsymbol{\gamma}$ is uniquely determined by $\boldsymbol{\gamma}(0)$ and $\tilde{\mathbf{t}}$, so $\boldsymbol{\gamma}$ is the unique solution.