A. What are the eigenvalues of X, Y, and Z? (Hint: First compute the trace and determinant. Then compute the eigenvalues from them.)

B. Beyond mere practice with one-qbit gates, this exercise plays a crucial role in quantum circuits that we study later in the course. Mermin proves this result in his book, but in a scattered and complicated way that ties into a bunch of cool math. Because I'm not trying to teach you a bunch of cool math, I favor the following simpler approach.

1. Let U be any diagonal one-qbit gate. Let $(\det U)^{1/2}$ be either of the two complex numbers that square to $\det U$. Prove, as explicitly as possible, that there exist diagonal one-qbit gates V and W such that
$$U = (\det U)^{1/2} V X V^* W X W^*.$$
2. Repeat part 1 of this problem, but with both occurrences of the word "diagonal" removed. (Hint: If you read the 2×2 unitary matrix part of our Complex Linear Algebra tutorial, you will find a fact that, in combination with part 1, greatly expedites this problem.)

C. This exercise develops a more refined picture of the set of one-qbit states, taking into account global phase changes.

1. Let $|\psi\rangle \in \mathbb{C}^2$ be any one-qbit state. Prove that there exist $t \in [0, 2\pi)$, $w \in [0, 2\pi)$, and $v \in [0, \pi]$ such that
$$|\psi\rangle = e^{it} \left[\begin{array}{c} \cos(v/2) \\ \sin(v/2)e^{iw} \end{array} \right]. \tag{1}$$
2. When $v = 0$, which popular state arises (up to global phase change)? When $v = \pi$, which popular state arises?
3. Assume that $v = 0$ and $v = \pi$ are the only cases where differing values of w and v produce indistinguishable states. Explain — intuitively, not rigorously — why the set of physically distinguishable one-qbit states forms a sphere. (Hint: Spherical coordinates. If you don't know them, then look them up.)
4. Consider the map $|\chi\rangle \mapsto |\chi_0|^2$. Intuitively, this map sends a one-qbit state to its resulting probability distribution over the classical states. (Notice that $|\chi_1|^2$ is determined by $|\chi_0|^2$ because they sum to 1.) Interpret this map geometrically, as a map from the sphere to another set.

D. Implement the `measurement` function below. (My implementation is six lines of code.) Submit your code in the same file as the exercises above — not a separate Python file. Include a short demonstration that your code works.

```
import math
import random
import numpy

# The classical one-qbit states.
ket0 = numpy.array([1 + 0j, 0 + 0j])
ket1 = numpy.array([0 + 0j, 1 + 0j])

# Write this function. Its input is a one-qbit state. It returns either ket0 or ket1.
def measurement(state):
    pass

# For large m, this function should print a number close to 0.64. (Why?)
def measurementTest345(m):
    psi = 0.6 * ket0 + 0.8 * ket1
    def f():
        if (measurement(psi) == ket0).all():
            return 0
        else:
            return 1
    acc = 0
    for i in range(m):
        acc += f()
    return acc / m
```