Here are the fundamental algebraic rules for tensor products of vectors in $\mathbb{C}^2$. (I meant to state them in lecture, but I might have forgotten.) For all $|\chi\rangle, |\phi\rangle, |\omega\rangle \in \mathbb{C}^2$ and all $\sigma \in \mathbb{C}$:

- Associativity of multiplication: $\sigma(|\chi\rangle \otimes |\phi\rangle) = (\sigma |\chi\rangle) \otimes |\phi\rangle = |\chi\rangle \otimes (\sigma |\phi\rangle)$.
- Distributivity: $(|\chi\rangle + |\psi\rangle) \otimes |\omega\rangle = |\chi\rangle \otimes |\omega\rangle + |\phi\rangle \otimes |\omega\rangle$.
- Distributivity: $|\chi\rangle \otimes (|\phi\rangle + |\omega\rangle) = |\chi\rangle \otimes |\phi\rangle + |\chi\rangle \otimes |\omega\rangle$.

Consequently, the expression for the tensor product in coordinates is

$$|\chi\rangle \otimes |\phi\rangle = \begin{bmatrix} \chi_0 \\ \chi_1 \end{bmatrix} \otimes \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} \chi_0\phi_0 \\ \chi_0\phi_1 \\ \chi_1\phi_0 \\ \chi_1\phi_1 \end{bmatrix} = \begin{bmatrix} \chi_0 |\phi\rangle \\ \chi_1 |\phi\rangle \end{bmatrix}.$$

Okay, now let's get started on your homework.

A. Is the following identity true? Prove or disprove that, for all $|\chi\rangle, |\phi\rangle, |\omega\rangle, |\psi\rangle \in \mathbb{C}^2$,

$$|\chi\rangle \otimes |\phi\rangle + |\omega\rangle \otimes |\psi\rangle = (|\chi\rangle + |\omega\rangle) \otimes (|\phi\rangle + |\psi\rangle).$$

B. Prove that, if $\| |\chi\rangle \| = 1$ and $\| |\phi\rangle \| = 1$, then $\| |\chi\rangle \otimes |\phi\rangle \| = 1$. (So the tensor product of one-qbit states is a legitimate two-qbit state.)

C. We now have three distinct notions of measurement for a two-qbit state: full measurement of the state, partial measurement of the first qbit, and partial measurement of the second qbit. Prove that measuring the first qbit and then immediately measuring the second qbit has the same overall effect as full measurement — that is, the same possible outcomes with the same probabilities. (A similar proof shows that measuring the second qbit and then the first qbit has the same overall effect. You are not asked to do that proof.)

D. In lecture I claimed that almost all two-qbit states $|\psi\rangle$ violate the unentanglement equation $\psi_{00}\psi_{11} - \psi_{01}\psi_{10} = 0$. Let's test this idea in Python using our budding `qc.py` library. After invoking `import qc`, invoking `qc.uniform(2)` will produce a uniformly random two-qbit state. (You are not expected to understand the details of how or why `uniform` works.) So, in principle at least, you can test my claim by generating a large number of states and, for each one, testing whether it satisfies the unentanglement equation.

Unfortunately, it's useless to test whether the equation holds exactly, because the computer's underlying floating-point calculations are inexact. So you need to consider *by how much* the equation is violated. In my solution, I inspected the deciles (0th, 10th, 20th, etc. percentiles) of the mismatch in the equation. Write a short report consisting of your test code, the output, and a couple of sentences about your findings.