You have 60 minutes.

No notes, books, calculators, computers, etc. are allowed.

Show all of your work, in as organized a manner as possible. Incorrect answers with solid work often earn partial credit. Correct answers without explanatory work rarely earn full credit.

Perform as much algebraic simplification as you can. Do not bother to do non-trivial arithmetic. Simplify expressions such as $\sin 30°$ but not expressions such as $\sin 35°$. Mark your final answer clearly.

Good luck.

A. What is so unusual or notable about the vector field $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle$?

B. In this problem, all functions are smooth. Let $\vec{a}(t)$ and $\vec{b}(t)$ two 2D curves, and let $\vec{c}(t,\epsilon) = \vec{a}(t) + \epsilon\vec{b}(t)$. You can think of $\vec{c}$ as a surface with parameters t and ϵ, or you can think of $\vec{c}$ as a family of curves — one curve for each value of ϵ. Let $f(t,x,y)$ be a function. By evaluating f along $\vec{c}$, we can think of f as a function of t and ϵ. What equations hold at the critical points of f, regarded as a function of t and ϵ?

C. This problem is about fluid flow in 2D. The velocity of the fluid is $\vec{F} = \langle F_1(x, y), F_2(x, y)\rangle$. Let D be a simply connected region (no holes) with smooth boundary curve ∂D oriented counterclockwise. Let $\vec{c}(t) = (x(t), y(t))$ be a counterclockwise parametrization of ∂D (for $a \leq t \leq b$) and $\vec{n} = \langle y'(t), -x'(t)\rangle$ the resulting outward-pointing normal vector field along ∂D. The *flux* of the fluid across ∂D is defined to be

$$\int_{\partial D} \vec{F} \cdot \frac{\vec{n}}{|\vec{n}|}\, ds = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{n}(t)\, dt.$$

Let div $\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$. Show that the flux equals $\iint_D \text{div}\vec{F}\, dA$.

D. Consider a particle of electric charge q_1 situated at the origin, and a second particle of charge q_2 at position $\vec{x} = (x, y, z)$. Coulomb's law says that the electric force on the second particle is $\vec{F} = -\nabla V$, where $V(\vec{x}) = kq_1q_2/|\vec{x}|$ and $k > 0$ is a constant. Compute the work performed by $\vec{F}$ while the second particle moves from $\vec{P} = (1, 1, 1)$ to $\vec{Q} = (3, 5, 9)$ along the curve $\vec{c}(t) = (1 + t, 1 + t^2, 1 + t^3)$ for $0 \leq t \leq 2$.

E. Let W be the region bounded by $z = 1 - y^2$, $y = x^2$, and the planes $z = 0$, $y = 1$. Sketch W and compute its volume.

F. Show that the minimum distance from the origin to a point on the plane $ax + by + cz = d$ is $\dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

G1. Fix $R_1 < R_2$, and let D be the annulus defined by $R_1^2 \leq x^2 + y^2 \leq R_2^2$. Compute the double integral $\iint_D \log(x^2 + y^2)\, dA$. (Of course, log is $\log_e$. By the way, $\int \log u \, du = u \log u - u + C$.)

G2. So what is $\iint_D \log(x^2 + y^2)\, dA$, where D is the disk of radius R centered at the origin?