A.A. We compute

$$\partial f/\partial x = 2xy - 3z, \quad \partial f/\partial y = x^2, \quad \partial f/\partial z = -3x + 4z^3.$$

A.B. The tangent plane has equation $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$, where $\vec{p} = \langle 2, 1, 1 \rangle$ is a point on the plane and $\vec{n}$ is a normal vector. Because gradients are perpendicular to level sets, we can use $\vec{n} = \nabla f(2, 1, 1) = \langle 1, 4, -2 \rangle$. So the plane is $x + 4y - 2z = 4$.

[Some students tried another approach: Solving for z as a function $z = g(x, y)$ whose graph is the surface, and then using the linear approximation to g at $(x, y) = (2, 1)$ to get the tangent plane. In principle, this approach should work. In practice, no one could pull it off.]

A.C. We can parametrize the line as $\vec{x}(t) = \vec{p} + t\vec{d}$, where $\vec{d} = \vec{n}$. So the line is $\vec{x}(t) = \langle 2, 1, 1 \rangle + t\langle 1, 4, -2 \rangle$.

B. We want the directional derivative of f in the direction

$$\vec{v} = \langle \cos 3\pi/4, \sin 3\pi/4 \rangle = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle.$$

The gradient of f is $\nabla f = \langle e^{-x^2-y^2}(-2x), e^{-x^2-y^2}(-2y) \rangle$. So the directional derivative is

$$\nabla f(1, 0) \cdot \vec{v} = \langle -2e^{-1}, 0 \rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = \sqrt{2}/e.$$

C. [It is recommended that you first diagram how the variables depend on each other. Your diagram should show that p depends on T and V_m, which depend on time t.] The chain rule says that

$$\frac{dp}{dt} = \frac{\partial p}{\partial T}\frac{dT}{dt} + \frac{\partial p}{\partial V_m}\frac{dV_m}{dt}.$$

We compute

$$\begin{aligned} \frac{\partial p}{\partial T} &= \frac{R}{V_m - b} - \frac{a}{V_m(V_m + b)}\left(-\frac{1}{2}\right)T^{-3/2}, \\ \frac{\partial p}{\partial V_m} &= RT(-1)(V_m - b)^{-2} - \frac{a}{\sqrt{T}}(-1)(V_m^2 + V_m b)^{-2}(2V_m + b). \end{aligned}$$

At the moment of interest, we have

$$\begin{aligned} \frac{\partial p}{\partial T} &= \frac{R}{1 - b} + \frac{a}{16(1 + b)}, \\ \frac{\partial p}{\partial V_m} &= -4R(1 - b)^{-2} + \frac{a}{2}(1 + b)^{-2}(2 + b), \\ \frac{dp}{dt} &= 0.3, \\ \frac{dT}{dt} &= 0.1. \end{aligned}$$

Therefore

$$\begin{aligned}\frac{dV_m}{dt} &= \frac{\frac{dp}{dt} - \frac{\partial p}{\partial T}\frac{dT}{dt}}{\frac{\partial p}{\partial V_m}} \\ &= \frac{0.3 - 0.1\left(\frac{R}{1-b} + \frac{a}{16(1+b)}\right)}{-4R(1-b)^{-2} + \frac{a}{2}(1+b)^{-2}(2+b)}.\end{aligned}$$

[By the way, if you are unfamiliar with the Redlich-Kwong equation of state, try setting $a = b = 0$ and expressing V_m as V/n. Then you might get an equation that you've seen in a chemistry class.]

D. We want to maximize $f(a,t) = (a - 9.8)t^2/2$ subject to the constraint that $g(a,t) = a^2 t = 100{,}000$. We proceed by Lagrange multipliers. First,

$$\nabla g(a,t) = \langle 2at, a^2 \rangle.$$

So $\nabla g = \vec{0}$ only where $a = 0$, which does not satisfy the constraint $g = 100{,}000$. Thus far we have detected no points of interest. Next,

$$\nabla f(a,t) = \langle t^2/2, (a - 9.8)t \rangle.$$

We need to solve this system of three equations in three variables a, t, λ:

$$\begin{aligned} t^2/2 &= \lambda 2at, \\ (a - 9.8)t &= \lambda a^2, \\ a^2 t &= 100{,}000. \end{aligned}$$

The third equation requires that a and t be non-zero. Then the first equation implies that $\lambda = t/(4a)$. Plugging that expression for λ into the second equation, we obtain $a - 9.8 = a/4$, which implies that $a = 4(9.8)/3$.

We have found just one point of interest, where $a = 4(9.8)/3$ (and t and λ have some specific values that we could compute if we had to). Based on the meaning of the problem, f must have a maximum at this point, but let's check that intuition. At the point of interest, $a \approx 13$, and the third equation tells us that

$$t = \frac{100{,}000}{a^2} > \frac{100{,}000}{200} = 500,$$

so

$$f(a,t) > (13 - 10)(500)^2/2 > (100)^2 = 10{,}000.$$

In comparison, the point $(1, 100{,}000)$ satisfies the constraint with $f(1, 100{,}000) < 0$, and the point $(100, 10)$ satisfies the constraint with

$$f(100, 10) = (100 - 9.8)(50) \approx (90)(50) = 4{,}500.$$

So f decreases away from $a = 4(9.8)/3$, and the altitude is maximized there.

[Because of an ... irregularity, the test time was cut from 60 minutes to 50 minutes, and students did not complete problem E. However, here are the answers for posterity.]

E.A. TRUE. [This statement is Clairaut's theorem.]

E.B. FALSE. [The gradient could be undefined. Consider for example $f(x, y) = \sqrt{x^2 + y^2}$.]

E.C. FALSE. [We did a counter-example in class, which was shaped like the roof of a house.]

E.D. TRUE. [This statement is the definition of continuity.]

E.E. FALSE. [For example, $x^2 - y^2$ has a saddle at $\vec{0}$.]

E.F. FALSE. [The max could occur at a boundary point.]