A.A. [Draw a picture.] The integral is $\int_1^2 \int_y^2 x^2 y \, dx \, dy$.

A.B. The integral is $\int_1^2 \int_1^x x^2 y \, dy \, dx$.

A.C. Let's compute

$$\begin{aligned} \int_1^2 \int_1^x x^2 y \, dy \, dx &= \int_1^2 \left[\frac{1}{2} x^2 y^2\right]_{y=1}^x \, dx \\ &= \int_1^2 \frac{1}{2} x^4 - \frac{1}{2} x^2 \, dx \\ &= \left[\frac{1}{10} x^5 - \frac{1}{6} x^3\right]_1^2 \\ &= \left(\frac{32}{10} - \frac{8}{6}\right) - \left(\frac{1}{10} - \frac{1}{6}\right). \end{aligned}$$

The rest is arithmetic. We compute

$$\left(\frac{32}{10} - \frac{8}{6}\right) - \left(\frac{1}{10} - \frac{1}{6}\right) = \frac{31}{10} - \frac{7}{6} = \frac{93}{30} - \frac{35}{30} = \frac{58}{30} = \frac{29}{15}.$$

B. The mass is the integral of the density. We proceed in spherical coordinates:

$$\begin{aligned} \int_0^{2\pi} \int_0^{\pi} \int_0^R A e^{-B\rho} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta &= A \left(\int_0^{2\pi} 1 \, d\theta\right) \left(\int_0^{\pi} \sin\phi \, d\phi\right) \left(\int_0^R \rho^2 e^{-B\rho} \, d\rho\right) \\ &= A \cdot 2\pi \cdot 2 \cdot \int_0^R \rho^2 e^{-B\rho} \, d\rho \\ &= 4\pi A \int_0^R \rho^2 e^{-B\rho} \, d\rho. \end{aligned}$$

To finish the problem, we need to integrate by parts twice. [Students who reached this point earned 10/12 of the credit.] Letting $u = \rho^2$ and $dv = e^{-B\rho} \, d\rho$, we have $du = 2\rho \, d\rho$, $v = -\frac{1}{B} e^{-B\rho}$, and

$$\begin{aligned} \int \rho^2 e^{-B\rho} \, d\rho &= -\frac{1}{B} \rho^2 e^{-B\rho} - \int -\frac{2}{B} \rho e^{-B\rho} \, d\rho \\ &= -\frac{1}{B} \rho^2 e^{-B\rho} + \frac{2}{B} \int \rho e^{-B\rho} \, d\rho. \end{aligned}$$

Then, letting $u = \rho$ and $dv = e^{-B\rho} \, d\rho$, we have $du = d\rho$, $v = -\frac{1}{B} e^{-B\rho}$, and

$$\begin{aligned} \int \rho e^{-B\rho} \, d\rho &= -\frac{1}{B} \rho e^{-B\rho} - \int -\frac{1}{B} e^{-B\rho} \, d\rho \\ &= -\frac{1}{B} \rho e^{-B\rho} - \frac{1}{B^2} e^{-B\rho} + C. \end{aligned}$$

Therefore

$$\begin{aligned} \int \rho^2 e^{-B\rho} \, d\rho &= -\frac{1}{B} \rho^2 e^{-B\rho} + \frac{2}{B} \left(-\frac{1}{B} \rho e^{-B\rho} - \frac{1}{B^2} e^{-B\rho}\right) + C \\ &= -e^{-B\rho} \left(\frac{1}{B} \rho^2 + \frac{2}{B^2} \rho + \frac{2}{B^3}\right) + C, \end{aligned}$$

and

$$\begin{aligned}\int_0^R \rho^2 e^{-B\rho}\, d\rho &= \left[-e^{-B\rho}\left(\frac{1}{B}\rho^2 + \frac{2}{B^2}\rho + \frac{2}{B^3}\right)\right]_0^R \\ &= -e^{-BR}\left(\frac{1}{B}R^2 + \frac{2}{B^2}R + \frac{2}{B^3}\right) - -e^0\left(0+0+\frac{2}{B^3}\right) \\ &= \frac{2}{B^3} - e^{-BR}\left(\frac{1}{B}R^2 + \frac{2}{B^2}R + \frac{2}{B^3}\right).\end{aligned}$$

So finally the answer is

$$4\pi A\left(\frac{2}{B^3} - e^{-BR}\left(\frac{1}{B}R^2 + \frac{2}{B^2}R + \frac{2}{B^3}\right)\right).$$

C. Working in cylindrical coordinates, the total mass of pollution up to altitude 1/4 km is

$$\begin{aligned}\int_0^{1/4}\int_0^{2\pi}\int_0^R Az^{-1/2} r\, dr\, d\theta\, dz &= A\left(\int_0^{1/4} z^{-1/2}\, dz\right)\left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^R r\, dr\right) \\ &= A\left[2z^{1/2}\right]_0^{1/4}\cdot 2\pi\cdot\left[\frac{1}{2}r^2\right]_0^R \\ &= A\left(2\cdot\frac{1}{2}\right)\cdot 2\pi\cdot\left(\frac{1}{2}R^2\right) \\ &= A\pi R^2.\end{aligned}$$

D. [Draw some pictures.] The desired integral is

$$\begin{aligned}&\int_0^3\int_{-z/3}^{2-z}\int_0^{y+z/3} z\, dx\, dy\, dz \\ =\ &\int_0^3\int_{-z/3}^{2-z} yz + \frac{1}{3}z^2\, dy\, dz \\ =\ &\int_0^3 \left[\frac{1}{2}y^2 z + \frac{1}{3}yz^2\right]_{y=-z/3}^{2-z} dz \\ =\ &\int_0^3 \left(\frac{1}{2}(2-z)^2 z + \frac{1}{3}(2-z)z^2\right) - \left(\frac{1}{2}\left(-\frac{z}{3}\right)^2 z + \frac{1}{3}\left(-\frac{z}{3}\right)z^2\right) dz \\ =\ &\int_0^3 2z - 2z^2 + \frac{1}{2}z^3 + \frac{2}{3}z^2 - \frac{1}{3}z^3 - \frac{1}{18}z^3 + \frac{1}{9}z^3\, dz \\ =\ &\int_0^3 2z - \frac{4}{3}z^2 + \frac{2}{9}z^3\, dz \\ =\ &\left[z^2 - \frac{4}{9}z^3 + \frac{1}{18}z^4\right]_0^3 \\ =\ &9 - \frac{4}{9}\cdot 27 + \frac{1}{18}\cdot 81 \\ =\ &\frac{18}{2} - \frac{24}{2} + \frac{9}{2} \\ =\ &\frac{3}{2}.\end{aligned}$$

[For posterity, here are all six orders of integration. Notice that some of them require us to break the region into two or three subregions. In Mathematica I have confirmed that they all produce the answer of 3/2, but still they might contain small mistakes.

$$
\begin{aligned}
& \int_0^3 \int_{-z/3}^{2-z} \int_0^{y+z/3} z \, dx \, dy \, dz \\
= & \int_0^3 \int_0^{2-2z/3} \int_{x-z/3}^{2-z} z \, dy \, dx \, dz \\
= & \int_0^2 \int_0^{3-3x/2} \int_{x-z/3}^{2-z} z \, dy \, dz \, dx \\
= & \int_0^2 \int_x^2 \int_0^{2-y} z \, dz \, dy \, dx \\
& + \int_0^2 \int_{3x/2-1}^{x} \int_{3x-3y}^{2-y} z \, dz \, dy \, dx \\
= & \int_{-1}^0 \int_{-3y}^{2-y} \int_0^{y+z/3} z \, dx \, dz \, dy \\
& + \int_0^2 \int_0^{2-y} \int_0^{y+z/3} z \, dx \, dz \, dy \\
= & \int_0^2 \int_0^y \int_0^{2-y} z \, dz \, dx \, dy \\
& + \int_0^2 \int_y^{2y/3+2/3} \int_{3x-3y}^{2-y} z \, dz \, dx \, dy \\
& + \int_{-1}^0 \int_0^{2y/3+2/3} \int_{3x-3y}^{2-y} z \, dz \, dx \, dy.
\end{aligned}
$$

Students who wrote down any of these six expressions earned 10/12 of the credit, even if they did no computation after that.]