1 Introduction

Many mobile phones and other portable computers feature devices called *accelerometers*, which enable them to measure their position and orientation in three-dimensional space. These accelerometers let a phone detect whether it's in portrait or landscape orientation, and they are used to control some video games.

This project explores how a specific kind of accelerometer — the microelectromechanical (MEMS) spring accelerometer — works. For simplicity, it treats the case of one accelerometer detecting motion in one dimension. Mathematically, there are four main ingredients:

- the relationship between position and acceleration, which is given by integrating twice,
- the relationship between acceleration and force, which is given by Newton's Law of Motion,
- the relationship between force and displacement in a spring, which is given by Hooke's Law, and
- the relationship between displacement and voltage, which requires some elementary electrical theory and one more integration.

I originally wrote this project as a Calculus II exercise at Duke University around 2007. Now, in Multivariable Calculus at Carleton College in 2021, it is a completely optional "enrichment activity" for students who are curious about such things.

2 Acceleration, Force, and Spring Displacement

Suppose we have two parallel metal plates. One is mounted on a spring and the other is fixed. At time $t = 0$, the spring is relaxed — neither compressed nor stretched — and the distance between the two plates is D. Imagine that the spring-mounted plate experiences some force and thus is moved (see figure).

If we can measure the change in distance ΔD between the plates, then we can infer the force using Hooke's Law,

$$F = -k\Delta D,$$

where k is a known constant measuring the stiffness of the spring. If we also know the mass m of the moving plate, then we can use Newton's Law

$$F = ma$$

to infer the acceleration.

Problem 1. Find a formula for the acceleration a experienced by the moving plate, in terms of ΔD, m, and k.

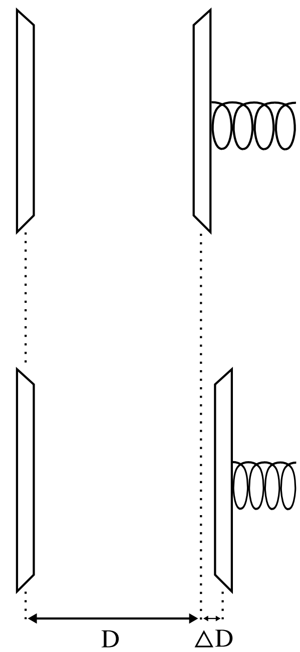

Figure 1: Two metal plates are situated at a distance D. Then the apparatus is moved, causing the spring to compress and the distance to change by ΔD.

3 Spring Displacement and Voltage

The tricky question is how to measure the displacement ΔD in the spring. This is where electricity comes in. One plate is given electrical charge $-Q$, and the other is given exactly the opposite charge, $+Q$. We keep the plates electrically insulated, so that the charges don't change over time. This arrangement is called a *parallel plate capacitor*. Let A be area of each plate, and let ϵ be the *electrical permittivity* of the material between them (a constant). Then throughout the region between the plates there is an electrical force of uniform strength

$$F = \frac{Q}{A\epsilon},$$

pointing from the positively-charged plate to the negatively-charged one. (To prove this fact from physical principles is not trivial, but we'll just assume it. We have to assume something.)

Voltage, denoted V, is defined as the work required to move a unit charge across the gap, from the positively-charged plate to the negatively-charged one. What is *work*? It is the integral of force applied across a distance. In this case, the voltage is the integral of the electrical force, running from the positively-charged plate to the negatively-charged plate.

Problem 2. By integrating from the positively-charged plate to the negatively-charged one (the distance between them being D), compute the voltage V between the two plates.

Problem 3. Compute how the voltage changes if the distance between the two plates changes by ΔD. That is, give a formula for ΔV in terms of the charge Q, permittivity ϵ, area A, and displacement ΔD.

4 Acceleration and Voltage

Before starting our accelerometer, we measure a baseline voltage V_0 in the circuit. We assume that the spring starts in a relaxed state, not compressed or stretched at all. If at some later time t we measure a voltage of $V(t)$, then we can compute the change in voltage $\Delta V = V(t) - V_0$, and thus the acceleration $a(t)$ at that time.

Problem 4. Using your formulas from the preceding sections, show that the acceleration a is proportional to the change ΔV in voltage: $a = K\Delta V$. What is the constant of proportionality K, in terms of the physical constants describing the device?

Problem 5. Give a final formula for the acceleration $a(t)$ at time t, in terms of the known constant K, the initial voltage V_0, and the voltage $V(t)$ measured at time t.

This last formula tells us the exact relationship between voltage (which we can measure) and acceleration (which we want to measure).

5 Acceleration, Velocity, and Position

A technique called Euler's method helps us deduce velocity from acceleration and displacement from velocity. Suppose that we wish to reconstruct y as a function of x, given only knowledge of the derivative $\frac{dy}{dx}$ and a starting point (x_0, y_0), such as $(2, 5)$. We first pick (or are given) a step size Δx. Then we make a table, like this:

Step k	x_k	y_k	$\frac{dy}{dx}$	$\Delta y = \frac{dy}{dx} \cdot \Delta x$
0	2	5		
1				
2				
3				
$\vdots$				

We find $\frac{dy}{dx}$ at the current point (x_0, y_0). This gives us a slope, which we multiply by the run Δx to get a rise Δy. We enter both of these in the first row of the table. Then we add Δx to x_0 to get x_1, and we add Δy to y_0 to get y_1. We enter x_1 and y_1 into the next line of the table. Then we repeat, finding $\frac{dy}{dx}$ at the new point (x_1, y_1), computing a new Δy, computing the next point (x_2, y_2), and so on.

We're going to use Euler's method to track the position $x(t)$ of our accelerometer as it moves around. We begin with a step size Δt, as well as a starting value for x *and* a starting value for the velocity v. At each time step, we measure the output voltage and convert it to an acceleration a using the formula from Problem 5. Then, since acceleration is the derivative of velocity, we multiply the acceleration by Δt to get a change Δv in velocity. Then, since velocity is the derivative of position, we multiply the velocity v by Δt to get a change Δx in position. We enter these in the first row of the table below. We also add Δt, Δv, and Δx to the current values of t, x, and v and enter the new values in the second line of the table. Then we repeat.

Step k	Time t_k	Voltage V_k	a_k	v_k	x_k	$\Delta v_k = a_k \cdot \Delta t$	$\Delta x_k = v_k \cdot \Delta t$
0							
1							
2							
3							
$\vdots$							

As part of your work for a mobile phone company, you are charged with incorporating an accelerometer made by another manufacturer into your company's next phone. The specifications on the accelerometer say that it outputs a baseline voltage of 1.5 volts when not being accelerated, and an additional 0.03 volts per 1 m/s^2 of acceleration. The specifications also indicate that the output voltage is updated 500 times per second, or once every 0.002 seconds. (These figures are based on the Analog Devices ADXL330 accelerometer, but they have been simplified somewhat.)

Problem 6. Using this information, determine K and V_0 and rewrite your formula from Problem 5 using those values. Check your formula with your instructor before proceeding.

Problem 7. Suppose that your phone begins at rest, with position x, velocity v, and acceleration a all assumed to be zero. The accelerometer outputs the following voltages over the first half second of time:

Time	0.000	0.100	0.200	0.300	0.400	0.500
Voltage	1.500	1.702	1.946	2.422	2.160	2.120

Make a table like the one above and use Euler's method to track the position of the accelerometer with a time step of $\Delta t = 0.100$. (Note: You could compute all of the accelerations, then all of the velocities, and then all of the positions, but it is more realistic to compute a single acceleration, velocity, and position before moving on to the next acceleration.)

6 Discussion

Over time, the approximate nature of Euler's method causes our record of the position x to drift away from the true position of the device. This is an inherent problem, but there are a few immediate improvements we could make, to reduce the error.

First, our accelerometer outputs a new voltage every 0.002 seconds, but we've been using a time step of $\Delta t = 0.100$. That is, we've been using only 2% of the accelerometer output. Switching to a smaller time step could lead to a more precise simulation.

Second, the doubled Euler's method we've been using implicitly assumes that both acceleration and velocity are constant during each time step. This causes the velocity and position to be updated at time step $t = t_k$ as

$$\begin{aligned} v(t) &= v_{k-1} + a_{k-1}(t - t_{k-1}), \\ x(t) &= x_{k-1} + v_{k-1}(t - t_{k-1}). \end{aligned}$$

However, if acceleration is constant, then the velocity should be linear in time, and the position should depend on time quadratically. That is, for times t between t_{k-1} and t_k we should have

$$\begin{aligned} v(t) &= v_{k-1} + a_{k-1}(t - t_{k-1}), \\ x(t) &= x_{k-1} + \int_{t_{k-1}}^{t} v(\tau)\, d\tau \\ &= x_{k-1} + \int_{t_{k-1}}^{t} v_{k-1} + a_{k-1}(\tau - t_{k-1})\, d\tau \\ &= x_{k-1} + v_{k-1}(t - t_{k-1}) + \frac{1}{2} a_{k-1}(t - t_{k-1})^2. \end{aligned}$$

Third, instead of assuming that the acceleration is constant during each time step we could linearly interpolate between one measured acceleration and the next. That is, between t_{k-1} and t_k we have

$$a(t) = a_{k-1} + \frac{a_k - a_{k-1}}{t_k - t_{k-1}} t.$$

Then velocity depends quadratically on time:

$$\begin{aligned} v(t) &= v_{k-1} + \int_{t_{k-1}}^{t} a(\tau)\, d\tau \\ &= v_{k-1} + \int_{t_{k-1}}^{t} a_{k-1} + \frac{a_k - a_{k-1}}{t_k - t_{k-1}} \tau\, d\tau \\ &= v_{k-1} + a_{k-1}(t - t_{k-1}) + \frac{1}{2} \frac{a_k - a_{k-1}}{t_k - t_{k-1}} (t^2 - t_{k-1}^2) \\ &= v_{k-1} - a_{k-1} t_{k-1} - \frac{1}{2} \frac{a_k - a_{k-1}}{t_k - t_{k-1}} t_{k-1}^2 + a_{k-1} t + \frac{1}{2} \frac{a_k - a_{k-1}}{t_k - t_{k-1}} t^2. \end{aligned}$$

Finally, position depends cubically on time:

$$\begin{aligned} x(t) &= x_{k-1} + \int_{t_{k-1}}^{t} v(\tau)\, d\tau \\ &= x_{k-1} + \int_{t_{k-1}}^{t} v_{k-1} - a_{k-1}t_{k-1} - \frac{1}{2}\frac{a_k - a_{k-1}}{t_k - t_{k-1}} t_{k-1}^2 + a_{k-1}\tau + \frac{1}{2}\frac{a_k - a_{k-1}}{t_k - t_{k-1}}\tau^2 \, d\tau \\ &= x_{k-1} + (v_{k-1} - a_{k-1}t_{k-1} - \frac{1}{2}\frac{a_k - a_{k-1}}{t_k - t_{k-1}} t_{k-1}^2)(t - t_{k-1}) \\ &\quad + \frac{1}{2}a_{k-1}(t - t_{k-1})^2 + \frac{1}{6}\frac{a_k - a_{k-1}}{t_k - t_{k-1}}(t - t_{k-1})^3. \end{aligned}$$

These formulas look complicated, but they are not difficult to program into a computer, and they lead to more accurate position-tracking than the constant-acceleration assumption.