1 Filling a Water Tower

Suppose that a particle of mass m is sitting on the plane x-y-plane, where $z = 0$. (Mass is measured in kg and distance in m.) To raise this particle against Earth's gravitational field to a given altitude $z > 0$ takes energy or *work*. The amount of work is mgz, where $g = 9.8$ m/s^2.

Now consider a solid, liquid, or gaseous body that occupies a region E of space and has density $\delta(x, y, z)$ (in kg/m^3). Assume that each particle in the body began in the $z = 0$ plane. The total work, to raise all of the particles in the body, from $z = 0$ to their current positions, equals $\iiint_E \delta g z \, dV$.

This concept can be used to calculate the energy needed to fill a water tower with water. Suppose that you have a cylindrical tank of radius R and height B. The tank is elevated, so that its bottom is at height A above the ground. The density of water is 1,000 kg/m^3.

A. Calculate the work needed to fill this water tower from the ground.

This amount of energy is important to local utilities, who literally pay to pump water into water towers. It is also relevant to people who design energy storage systems. For example, how do you save excess solar energy captured during the day, so that you can release it during the night? One idea is to pump water into towers during the day, then let it drain during the night, powering dynamos that generate electricity.

2 Making Hawaii

Geologists use similar concepts to understand how energy is distributed among geological processes. For example, along the San Andreas fault, the tectonic plates underlying North America and the Pacific Ocean grind against each other with tremendous kinetic energy. Any energy that goes into permanent topographic change (mountain building) is energy that's not available for earthquakes.

In that spirit, let's talk about the island of Hawaii, which is one enormous mountain. Let's focus on the part above sea level. It is a cone of height of 4,200 m and radius 58,000 m. Its density is a constant 3,000 kg/m^3.

B. How much work did it take to raise Hawaii from sea level?

The foregoing problem is much simpler than reality. Hawaii is not really a cone, and its density is not constant. Most of Hawaii is under sea level, but we're ignoring all of that, even though it is geologically relevant. The acceleration of Earth's gravitational field is not a constant 9.8 m/s^2 over regions as large as Hawaii. In particular, the under-sea part of the island noticeably alters the gravitational field experienced by the over-sea part.

3 Where is the Electron?

In quantum theory, a particle is described by a *wave function* ψ, and the probability of finding the particle in a region E of space is $\iiint_E |\psi|^2 \, dV$. In particular, the wave function for the $1s$ state of an electron in a hydrogen atom is $\psi(\rho) = (\pi a^3)^{-1/2} e^{-\rho/a}$, where ρ is the distance to the nucleus (at the origin), and $a = 5.3 \cdot 10^{-11}$ m is a constant called the *Bohr radius*.

C. Compute the probability of finding the electron at a distance of R or less from the nucleus.

4 Mass of the Atmosphere

In this problem, let's assume that the Earth is a spherical ball of radius $R = 6{,}371$ km. Measurements indicate that the density of the atmosphere drops off exponentially with altitude. To be precise, at an altitude of h km above the Earth's surface, the atmosphere has density $\delta(h) = ae^{-bh}$ kg/km^3, where $a = 1.225 \cdot 10^9$ and $b = 0.13$. There is no clear boundary between the atmosphere and space; the atmosphere just keeps getting thinner and thinner.

D. Calculate the total mass of Earth's atmosphere.