1 Definitions

Recall that, if $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar field on $\mathbb{R}^3$, then the gradient of f is the vector field

$$\text{grad } f = \nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right\rangle$$

on $\mathbb{R}^3$. Now suppose that $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field on $\mathbb{R}^3$. We define the *curl* of $\vec{F}$ to be the vector field

$$\text{curl } \vec{F} = \left\langle \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right\rangle$$

and the *divergence* of $\vec{F}$ to be the scalar field

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.$$

Here's a helpful mnemonic device. Define the "vector" ∇ to be

$$\nabla = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\rangle.$$

I put "vector" in quotation marks because this thing is not a vector — or at least not a vector of any kind that we've studied. If you ignore such philosophical scruples and compute blindly, then you discover that the gradient ∇f is indeed the vector ∇ scaled on the right by the scalar-valued thing f:

$$\nabla f = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\rangle f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right\rangle.$$

Moreover, you discover that the curl and divergence are cross and dot products:

$$\text{curl } \vec{F} = \nabla \times \vec{F}, \quad \text{div } \vec{F} = \nabla \cdot \vec{F}.$$

The curl and divergence are frequently written in this notation, so get used to it.

Each of these concepts has a geometric meaning. The gradient ∇f tells you, at each $\vec{x}$, how to climb the hill defined by f as quickly as possible there. The curl $\nabla \times \vec{F}$ tells you, at each $\vec{x}$, how much $\vec{F}$ rotates or curls there. The divergence $\nabla \cdot \vec{F}$ tells you, at each $\vec{x}$, whether $\vec{F}$ spreads out or comes together there. See the companion Mathematica notebook.

These concepts also have many applications. For example, almost everything that we encounter in daily life is a consequence of electromagnetism, which (in its pre-quantum theory) is governed by Maxwell's equations. In the simplest conditions, these equations relate a time-dependent vector field $\vec{E}$ for electricity and a time-dependent vector field $\vec{H}$ for magnetism as follows.

$$\text{div } \vec{E} = 0, \quad \text{div } \vec{H} = 0, \quad \text{curl } \vec{E} = -\frac{\partial}{\partial t}\vec{H}, \quad \text{curl } \vec{H} = \frac{\partial}{\partial t}\vec{E}.$$

2 In other dimensions

The gradient and the divergence generalize simply to all dimensions. In n dimensions, define

$$\nabla = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle.$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar field and $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, then define

$$\text{grad } f = \nabla f = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle, \quad \text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

The composition of the divergence and the gradient is another operation, called the *Laplacian*, which we haven't discussed yet. It has various popular notations:

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \text{div}(\text{grad } f) = \frac{\partial^2 F_1}{\partial x_1^2} + \cdots + \frac{\partial^2 F_n}{\partial x_n^2}.$$

To clarify, the Laplacian of a scalar field f is a scalar field Δf. The Laplacian arises in mathematical models of heat conduction, fluid dynamics, groundwater diffusion, and other scientific problems. The two-dimensional version plays a prominent role in complex analysis (calculus with complex numbers).

The curl does not generalize to all dimensions, because it is related to the cross product, which is specific to three dimensions. However, people sometimes talk about the curl of a two-dimensional vector field $\vec{F}(x_1, x_2) = \langle F_1(x_1, x_2), F_2(x_1, x_2) \rangle$ by regarding it as a three-dimensional vector field $\vec{G}$ with nothing happening in the third dimension:

$$\vec{G}(x_1, x_2, x_3) = \langle F_1(x_1, x_2), F_2(x_1, x_2), 0 \rangle.$$

If you interpret the two-dimensional $\vec{F}$ as a three-dimensional $\vec{G}$ in this way, then you can compute

$$\text{curl } \vec{F} = \text{curl } \vec{G} = \left\langle 0, 0, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right\rangle.$$

The first two components of curl $\vec{F}$ are ultra-boring, but the third component of curl $\vec{F}$ is ripe with meaning. In which big theorem have you seen it? And when have you seen it equal zero?

3 de Rham cohomology

Let's return to three dimensions specifically. Let $\mathcal{S}$ be the set of all scalar fields $f : \mathbb{R}^3 \to \mathbb{R}$ that are smooth — meaning, every derivative that you can imagine, such as $f_{x_1 x_3 x_3 x_2 x_1 x_1 x_1 x_2 x_2}$, is continuous. Let $\mathcal{V}$ be the set of all vector fields $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ that are smooth in each of the three components. Then the gradient, curl, and divergence fit into this diagram of functions:

$$\mathcal{S} \xrightarrow{\text{grad}} \mathcal{V} \xrightarrow{\text{curl}} \mathcal{V} \xrightarrow{\text{div}} \mathcal{S}.$$

In other words, the gradient of a smooth scalar field is a smooth vector field, the curl of a smooth vector field is a smooth vector field, and the divergence of a smooth vector field is a smooth scalar field.

Now here's an important fact: For any scalar field f, curl(grad f) $= \vec{0}$. (Prove so algebraically. See the companion Mathematica notebook to compute concrete examples.) So if $\vec{F} = \text{grad } f$, then curl $\vec{F} = \vec{0}$. In other words, if curl $\vec{F} \neq \vec{0}$, then $\vec{F}$ is not conservative.

Here's another important fact: For any vector field $\vec{F}$, div(curl $\vec{F}$) $= 0$. (Prove so algebraically, or visit the Mathematica notebook.) In other words, if div $\vec{G} \neq 0$, then $\vec{G}$ is not the curl of any vector field $\vec{F}$.

Are the converses to these statements true? If $\vec{F}$ has curl $\vec{0}$, then must $\vec{F}$ be conservative? If $\vec{G}$ has divergence 0, then must $\vec{G}$ be a curl? Also, if f has gradient $\vec{0}$, then what does that say about f? The answers depend on whether there are holes in the domains of f, $\vec{F}$, and $\vec{G}$.

A "point hole" in the domain of $\vec{G}$ allows for the possibility that div $\vec{G} = 0$ but $\vec{G} \neq \text{curl } \vec{F}$. The crucial example is $\vec{G} = \vec{x}/\left|\vec{x}\right|^{3/2}$ (or any scalar multiple of the Newtonian gravitational force field). It is undefined at the point $\vec{0}$. Everywhere else, its divergence is 0. However, there is no $\vec{F}$ defined everywhere $\vec{G}$ is defined, such that curl $\vec{F} = \vec{G}$. (Briefly, the argument is: The flux of $\vec{G}$ across the unit sphere is positive, but the flux of curl $\vec{F}$ must be 0 by Stokes's theorem below.)

A "line hole" in the domain of $\vec{F}$ allows for the possibility that curl $\vec{F} = \vec{0}$ but $\vec{F} \neq \text{grad } f$. The crucial example is $\vec{F} = \langle -x_2, x_1, 0\rangle/(x_1^2 + x_2^2)$. This vector field is undefined along the line $x_1 = x_2 = 0$. Everywhere else, its curl is $\vec{0}$. We studied the two-dimensional version in homework.

A "plane hole" in the domain of f allows for the possibility that grad $f = \vec{0}$ but f is not constant. For example, define

$$\begin{cases} f(\vec{x}) = 31 & \text{if } x_3 > 0 \\ f(\vec{x}) = -4 & \text{if } x_3 < 0 \end{cases}.$$

This scalar field is undefined along the plane $x_3 = 0$. Everywhere else, its gradient is $\vec{0}$.

In summary, the existence of divergence-0 vector fields that are not curls, curl-$\vec{0}$ vector fields that are not gradients, or gradient-$\vec{0}$ scalar fields that are not constant tells us something about the holes in the underlying domain. There is an interplay between delicate, finicky questions of calculus and simple, crude questions of topology (how spaces connect up on themselves). In higher mathematics, these ideas get systematized in a concept called *de Rham cohomology*.

4 Stokes's theorem

The table below summarizes five big theorems of calculus, three of which we have studied.

dimension	straight version	curved version
1	$\int_{[a,b]} F'(x_1)\, dx_1 = F(b) + -F(a)$.	$\int_C \nabla f \cdot d\vec{x} = f(\vec{x}(b)) + -f(\vec{x}(a))$.
2	$\iint_D \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\, dx_1\, dx_2 = \int_C \vec{F} \cdot d\vec{x}$.	$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{x}$.
3	$\iiint_E \text{div } \vec{G}\, dx_1\, dx_2\, dx_3 = \iint_S \vec{G} \cdot d\vec{S}$.	

The upper-left theorem is the fundamental theorem of calculus. The upper-right theorem is the fundamental theorem of calculus for line integrals. Both the interval $[a, b]$ and the curve C are one-dimensional. The interval $[a, b]$ is essentially a curve C that happens to be straight.

The middle row of the table features Green's theorem on the left and Stokes's theorem (Section 16.8) on the right. Both the planar region D and the surface S are two-dimensional. The planar region D is essentially a surface S that happens to be straight.

The third row of the table features the divergence theorem (Section 16.9) on the left. It concerns an integral over a solid three-dimensional region E, and another integral over the surface S that bounds E. The right side of the third row is left blank, because the corresponding theorem over three-dimensional "curved solids" is rarely taught in calculus courses.

All five theorems listed in the table follow a single pattern. In each equation, the right integrand is a function, and the left integrand is some kind of derivative of that function. In each equation, the left integral is over an n-dimensional space, and the right integral is over its $(n-1)$-dimensional boundary. (The boundary of a one-dimensional space is a zero-dimensional space, which is a set of discrete points. Those points are assigned positive and negative signs according to a certain rule. An integral over a set of discrete points is simply a sum.)

The table doesn't end at three dimensions. There is a way to systematize everything that we've learned, so that it extends to curved (or straight) spaces of all dimensions. The integrand and the $dx_1\ dx_2\ \ldots$ get combined into an object called a *differential form*, denoted something like ω. A general derivative of this ω, denoted $d\omega$, can be rigorously defined. The boundary of a space X is denoted ∂X. Then, under appropriate conditions, we obtain a single, grand *Stokes's theorem* that works in all dimensions:

$$\int_X d\omega = \int_{\partial X} \omega.$$

We can't possibly learn all of this stuff in Math 211. Sometimes it's taught in the special topics course Math 295. Sometimes comps projects are done on it. I first learned it in mathematics graduate school. So why do I mention it in Math 211? To emphasize that there is a pattern to the five theorems. To give you a hint that, if you go on in mathematics, you will find the subject getting more abstract but, surprisingly, simpler.