To optimize $f(x,y)$ subject to the constraint that $g(x,y)=c$, examine all points (x,y) along $g(x,y)=c$ such that ∇f is parallel to ∇g. In other words, do these two tasks:

Examine all (x,y) where $g(x,y)=c$ and $\nabla g(x,y)=\vec{0}$.

Examine all (x,y,λ) such that $g(x,y)=c$ and $\nabla f(x,y)=\lambda\nabla g(x,y)$.

Problem A: Optimize $f(x,y)=e^{xy}$ subject to $x^3+y^3=16$.

Problem B: Optimize $f(x,y)=x$ subject to $y^2+x^4-x^3=0$.

To optimize $f(x,y,z)$ subject to the constraint that $g(x,y,z)=c$, examine all points (x,y,z) along $g(x,y,z)=c$ such that ∇f is parallel to ∇g. In other words:

Examine all (x,y,z) where $g(x,y,z)=c$ and $\nabla g(x,y,z)=\vec{0}$.

Examine all (x,y,z,λ) such that $g(x,y,z)=c$ and $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$.

Problem C: Find the points on the unit sphere $x^2+y^2+z^2=1$ that are closest to and farthest from the point $(3,2,1)$. Use Lagrange multipliers, and then check your answer using a simpler argument involving vectors.

To optimize $f(x,y,z)$ subject to the constraints that $g(x,y,z)=c$ and $h(x,y,z)=d$:

Examine all (x,y,z) where $g=c$, $h=d$, and ∇g is parallel to ∇h.

Examine all (x,y,z,λ,μ) where $g=c$, $h=d$, and $\nabla f=\lambda\nabla g+\mu\nabla h$.

Problem D: Optimize $f(x,y,z)=x+2y+3z$ on the intersection of the cylinder $x^2+y^2=1$ and the plane $x-y+z=1$.

Problem E: Do problem D in a different way, from earlier in our course: Parametrize the ellipse where the two constraints meet, to get x, y, and z as functions of t. Then optimize f as a function of t. Do you get the same answer?