Definition: The point (a, b) is a *critical point* of the function $f(x, y)$ if two conditions are satisfied: $f_x(a, b)$ is zero or non-existent, and $f_y(a, b)$ is zero or non-existent.

Theorem: If f has a local optimum at (a, b), then (a, b) is a critical point.

Definition: The *discriminant* of f is $D = f_{xx}f_{yy} - f_{xy}f_{yx}$.

Theorem (Second Derivative Test): Suppose that (a, b) is a critical point of f, and $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ are all continuous at (a, b). Then:

A. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local minimum.
B. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local maximum.
C. If $D(a, b) < 0$, then (a, b) is a saddle.
D. (If $D(a, b) = 0$, then the test is inconclusive.)

Theorem: If f is continuous on a closed, bounded region R, then f achieves a global maximum and a global minimum on R, and they occur at the critical points of f or at the boundary of R.

Problem A: In each sub-problem, find the critical points. Characterize them as minima, maxima, saddles, or unknown.

1. $f(x, y) = x^2 + y^2$. Draw a picture.
2. $f(x, y) = x^4 + y^4$. Draw a picture.
3. $f(x, y) = x^2 - 12xy + y$.
4. $f(x, y) = x^2$. Draw a picture.
5. $f(x, y) = 3xy^2 - x^3$. By the way, this surface is called the monkey saddle.

Problem B: The nose of a rocket is shaped like the region bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$. Scientists are trying to equip the rocket with a box-shaped instrument. They'd like the instrument to have the largest volume possible, subject to the constraint that it must fit into the nose. What's the solution?