A.A. The law of the unconscious statistician says that

$$E(X^4) = \int_{-\infty}^{\infty} x^4 f_X(x)\, dx = \int_0^{\infty} x^4 \lambda e^{-\lambda x}\, dx.$$

A.B. First, notice that the support of Y is $(0, \infty)$. Second,

$$F_Y(y) = P(Y \leq y) = P(X^4 \leq y) = P(X \leq y^{1/4}) = F_X(y^{1/4}).$$

Then, differentiating this equation with respect to y, we obtain

$$f_Y(y) = f_X(y^{1/4}) \frac{d}{dy}(y^{1/4}) = \lambda e^{-\lambda y^{1/4}} \frac{1}{4} y^{-3/4}.$$

Therefore

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^{\infty} \frac{1}{4} y^{1/4} \lambda e^{-\lambda y^{1/4}}\, dy.$$

A.C. Yes, the two integrals are equal. After all, Y and X^4 are the same random variable, so they must have the same expectation. Further, if we plug $y = x^4$ and $dy = 4x^3\, dx$ into the integral for $E(Y)$, and change the bounds from $(0, \infty)$ to $(0, \infty)$, then we recover the integral for $E(X^4)$.

B.A. By linearity of expectation,

$$E(S_n) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \lambda = n\lambda.$$

B.B. Because the X_i are independent, their variances add:

$$\mathrm{Var}(S_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = \sum_{i=1}^{n} \lambda = n\lambda.$$

B.C. The expectation is unchanged. The variance should incorporate covariance terms:

$$\begin{aligned}
\mathrm{Var}(S_n) &= \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{i<j} \mathrm{Cov}(X_i, X_j) \\
&= n\lambda + 2 \sum_{i<j} \left(E(X_i X_j) - E(X_i)E(X_j)\right) \\
&= n\lambda + 2 \sum_{i<j} \lambda/2 \\
&= n\lambda + 2 \binom{n}{2} \lambda/2 \\
&= \left(n + \binom{n}{2}\right) \lambda \\
&= \frac{n^2 + n}{2} \lambda.
\end{aligned}$$

C. TRUE. [This statement is the definition of the marginal distribution of X.]

D. First we do a little algebra:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,dx &= \int_{-\infty}^{\infty} e^{-\left(\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2-\left(\frac{b}{2\sqrt{a}}\right)^2+c\right)}\,dx \\
&= e^{\left(\frac{b}{2\sqrt{a}}\right)^2-c}\int_{-\infty}^{\infty} e^{-a\left(x+\frac{b}{2a}\right)^2}\,dx \\
&= e^{\left(\frac{b}{2\sqrt{a}}\right)^2-c}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\,dx,
\end{aligned}$$

where $\mu = -\frac{b}{2a}$ and $\sigma^2 = \frac{1}{2a}$. The integrand on the right is the un-normalized density for a random variable that is normally distributed with mean μ and variance σ^2. So the integral on the right is $\sqrt{2\pi\sigma^2} = \sqrt{\pi/a}$, and the answer to the entire problem is

$$e^{\left(\frac{b}{2\sqrt{a}}\right)^2-c}\sqrt{\frac{\pi}{a}}.$$

E. Let $\sigma^2 = 4 \cdot 10^{-6}$ and let μ be the true answer. So the results are distributed as $X \sim \text{Norm}(\mu, \sigma^2)$. We are told that $P(X < 0) = 0.025$. We know that 95% of the normal distribution's mass is within 2σ of μ. So 2.5% of the mass is to the left of $\mu - 2\sigma$. Therefore our guess for the true answer, based on the information given, should be $\mu = 2\sigma = 4 \cdot 10^{-3}$. [By the way, this problem is not realistic. The physicist would estimate the true answer by taking the average of her experimental results.]

F.A. Let time be measured in days. Then λ is the expected number of purchases per day, so $\lambda = 5/7$. [If instead time were measured in weeks, then we would have $\lambda = 5$.]

F.B. Let $X \sim \text{Expo}(\lambda)$ be the time in days until his next purchase. Then $E(X) = 1/\lambda = 7/5$.

F.C. Let $Y \sim \text{Pois}(7\lambda)$ be the number of purchases in the next 7 days. Then

$$\begin{aligned}
P(Y \geq 3) &= 1 - P(Y=0) - P(Y=1) - P(Y=2) \\
&= 1 - e^{-7\lambda}\left(1 + 7\lambda + (7\lambda)^2/2\right) \\
&= 1 - e^{-5}\left(1 + 5 + 25/2\right).
\end{aligned}$$

F.D. Let $Z \sim \text{Pois}(30\lambda)$ be the number of purchases in the next 30 days. Then $E(Z) = 30\lambda = 150/7$.

G. FALSE. [We computed an example in class where $X \sim \text{Unif}(-1,1)$ and $Y = X^2$.]

H. By the law of total probability and the definition of conditional probability,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\, dx.$$

We are told that $f_X(x) = xe^{-x^2/2}$ for x in $(0, \infty)$ and $f_{Y|X}(y|x) = x^2 e^{-x^2 y}$ for y in $(0, \infty)$. Therefore

$$\begin{aligned} f_Y(y) &= \int_0^{\infty} x^2 e^{-x^2 y} x e^{-x^2/2}\, dx \\ &= \int_0^{\infty} x^3 e^{-x^2(y+1/2)}\, dx. \end{aligned}$$

Using integration by parts with $u = x^2$ and $dv = xe^{-x^2(y+1/2)}\, dx$, hence $du = 2x\, dx$ and $v = \frac{-1}{2(y+1/2)} e^{-x^2(y+1/2)} + C$, we obtain

$$f_Y(y) = \left[\frac{-x^2}{2(y+1/2)} e^{-x^2(y+1/2)} \right]_{x=0}^{x=\infty} - \int_0^{\infty} \frac{-2x}{2(y+1/2)} e^{-x^2(y+1/2)}\, dx.$$

The first term on the right side evaluates to $0 - 0$. [The first 0 is from using l'Hopital's rule twice. I'll omit the details.] So we have

$$\begin{aligned} f_Y(y) &= \int_0^{\infty} \frac{2x}{2(y+1/2)} e^{-x^2(y+1/2)}\, dx \\ &= \left. \frac{-1}{2(y+1/2)^2} e^{-x^2(y+1/2)} \right]_{x=0}^{x=\infty} \\ &= 0 - \frac{-1}{2(y+1/2)^2} e^0 \\ &= \frac{1}{2(y+1/2)^2}. \end{aligned}$$

The support of Y is $(0, \infty)$.