A. Yes. In general, $V(X+Y) = V(X) + V(Y) + 2\text{Cov}(X, Y)$. To say that X and Y are uncorrelated is to say that $\text{Cov}(X, Y) = 0$. Therefore $V(X+Y) = V(X) + V(Y)$.

[Some students answered "no", because the variables are not assumed to be independent. But it's enough for them to be uncorrelated, by the above argument. In summary, here are two true statements: If X and Y are independent, then they are uncorrelated. And X and Y are uncorrelated if and only if $V(X+Y) = V(X) + V(Y)$.]

B.A. [This problem is part of Theorem 5.3.1 in our textbook. It was not assigned as reading or discussed in class.] We proceed by our usual transformation method:

$$F_Y(y) = P(Y \leq y) = P(F^{-1}(X) \leq y) = P(X \leq F(y)).$$

(The inequality does not flip, because F is an increasing function.) And then

$$P(X \leq F(y)) = F_X(F(y)).$$

[Up to here the problem is "standard". Now we have to think a little more creatively.] Notice that $0 \leq F(y) \leq 1$ for all y. Also, $F_X(x) = x$ for $0 \leq x \leq 1$. Therefore

$$F_X(F(y)) = F(y).$$

We conclude that the CDF of Y is F.

B.B. [This problem is Example 5.3.3 in our textbook. It was not assigned as reading or discussed in class.] For the sake of clarity, let Y be Juanita's score in the exam. Then $F(Y)$ is the probability that another randomly chosen student scores lower than Juanita. In other words, it is Juanita's *quantile* among the test-takers. Also, notice that $F(Y) = F(F^{-1}(X)) = X$ is uniform. So quantiles are uniformly distributed among test-takers. That makes sense. [Why?]

C.A. The expectation of X is

$$E(X) = \sum_x x P(X = x) = 1 \cdot 20/38 + -1 \cdot 18/38 = 2/38.$$

C.B. The expectation of X^2 is

$$E(X^2) = \sum_x x^2 P(X = x) = 1 \cdot 20/38 + 1 \cdot 18/38 = 1.$$

Therefore $V(X) = E(X^2) - (E(X))^2 = 1 - (2/38)^2$, and $SD(X) = \sqrt{1 - (2/38)^2}$.

C.C. Let $X_1, \ldots, X_n$ be the casino's earnings in n independent runs of the game, so that $S_n = X_1 + \cdots + X_n$. By linearity of expectation,

$$E(S_n) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = n \cdot 2/38.$$

C.D. Because the variables $X_1, \ldots, X_n$ defined above are independent, their variances add like this:

$$V(S_n) = V(X_1 + \cdots + X_n) = V(X_1) + \cdots + V(X_n) = n(1 - (2/38)^2).$$

Thus $SD(S_n) = \sqrt{n} \cdot \sqrt{1 - (2/38)^2}$.

[To understand the practical importance of this problem, think about these remarks: First, $SD(X)$ is much larger than $E(X)$. So the casino has a good chance of losing money on any one run of the game. Second, when n is large, $SD(S_n)$ is much smaller than $E(X_n)$. So the casino has a good chance of earning money each week, month, year, etc. That is, the casino is reliably profitable, even though the individual runs are not.]

D.A. In a Poisson process of rate λ, the number of occurrences in the interval $[0, \ell]$ (or any other interval of length ℓ) is $X \sim \text{Pois}(\lambda\ell)$.

D.B. The probability of a recombination is

$$P(X \text{ is odd}) = \sum_{k \text{ odd}} P(X = k) = \sum_{k=0}^{\infty} e^{-\lambda\ell} \frac{(\lambda\ell)^{2k+1}}{(2k+1)!}.$$

D.C. [This problem is not easy, but it's essentially Section 8.9 Exercise 11, which was assigned as homework.] Well,

$$\begin{aligned} P(X \text{ is even}) - P(X \text{ is odd}) &= \sum_{k=0}^{\infty} e^{-\lambda\ell} \frac{(\lambda\ell)^{2k}}{(2k)!} - \sum_{k=0}^{\infty} e^{-\lambda\ell} \frac{(\lambda\ell)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k e^{-\lambda\ell} \frac{(\lambda\ell)^k}{k!} \\ &= e^{-\lambda\ell} \sum_{k=0}^{\infty} \frac{(-\lambda\ell)^k}{k!} \\ &= e^{-\lambda\ell} e^{-\lambda\ell} \\ &= e^{-2\lambda\ell}. \end{aligned}$$

Meanwhile, $P(X \text{ is even}) + P(X \text{ is odd}) = 1$. Therefore $P(X \text{ is odd}) = \frac{1}{2}(1 - e^{-2\lambda\ell})$.

E.A. First, X can take on only positive values, while Y can take on any value. So the support of the joint PDF is the "right half plane" $\{(x, y) : x > 0\}$. Second, we are told that

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2}.$$

By the definition of conditional probability density, the joint PDF is

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2} x e^{-x^2/2}$$

on its support.

E.B. The marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2} x e^{-x^2/2}\,dx.$$

E.C. No, X and Y are not independent. Intuitively, knowing the value of X gives us information about the value of Y. For example, if X takes on value 1,000, then Y will probably take on a value near 1,000. More rigorously, if X and Y were independent then we'd have $f_{Y|X}(y|x) = f_Y(y)$. But the former is a non-trivial function of both x and y, while the latter is a function of y alone. So they cannot be equal.