Let $f : \mathbb{N} \to \mathbb{N}$ be a function that grows without bound. That is, $\lim_{n\to\infty} f(n) = \infty$. In the first three problems below, you prove that $\mathcal{O}(2^{f(n)})$ is a proper subset of $2^{\mathcal{O}(f(n))}$.

A. Give rigorous definitions of $\mathcal{O}(2^{f(n)})$ and $2^{\mathcal{O}(f(n))}$. (You might want to do the next two problems, before deciding that you're done with problem A.)

B. Prove that if g is $\mathcal{O}(2^{f(n)})$ then g is $2^{\mathcal{O}(f(n))}$.

C. Prove that there is a g in $2^{\mathcal{O}(f(n))}$ that is not in $\mathcal{O}(2^{f(n)})$.

Earlier in our course — maybe on Day 15? — we described a Turing machine for testing whether a given directed graph was in fact a connected undirected graph.

D. What are the time complexity and space complexity of that Turing machine? Analyze them in detail, and state your answers using $\mathcal{O}$ notation. Actually, give two answers for each: one in terms of the input size n, and one in terms of the number m of nodes in the graph.