Is NP closed under complementation? Nobody knows, but the common suspicion is that NP is not closed under complementation. Problems A and B seem to be proofs that NP is closed under complementation. Explain what is wrong with each of them. (The proofs are extremely similar in their texts but quite different in their logical errors.)

A. Let $A \in NP$. Then there exists an NTM N and natural number k such that $L(N) = A$ and the running time of N is $\mathcal{O}(n^k)$. Define a TM M that, on input w, runs N on w and outputs the opposite of what N outputs. Then $L(M) = L(N)^c = A^c$, and the running time of M is $\mathcal{O}(n^k)$. So $A^c \in NP$.

B. Let $A \in NP$. Then there exists an NTM N and natural number k such that $L(N) = A$ and the running time of N is $\mathcal{O}(n^k)$. Define an NTM M that, on input w, runs N on w and outputs the opposite of what N outputs. Then $L(M) = L(N)^c = A^c$, and the running time of M is $\mathcal{O}(n^k)$. So $A^c \in NP$.

For problems C and D, recall that

$$\mathit{CLIQUE} = \{\langle G, k\rangle : G \text{ is an undirected graph, } k \geq 1, \text{ and } G \text{ contains a } k\text{-clique}\}.$$

Also, for any $k \geq 1$, let

$$\mathit{CLIQUE}_k = \{\langle G\rangle : G \text{ is an undirected graph that contains a } k\text{-clique}\}.$$

In class, we will soon learn that CLIQUE is NP-complete. Without going into details, this fact implies that if $\mathit{CLIQUE} \in P$, then $P = NP$. The popular belief is that $P \neq NP$ and hence $\mathit{CLIQUE} \notin P$.

C. Show that $\mathit{CLIQUE}_k \in P$ for all k. (For the sake of problem D, it might help if you try to pin down your running time fairly precisely. By the way, the $k = 3$ case is Problem 7.9 in our textbook.)

D. Explain how it's possible that $\mathit{CLIQUE}_k \in P$ for all k, but $\mathit{CLIQUE} \notin P$. In other words, explain why someone might think that $(\forall k\ \mathit{CLIQUE}_k \in P) \Rightarrow \mathit{CLIQUE} \in P$, and why that argument can't be completed.