A. Yes, A is regular. A DFA with three states suffices. [Fill in the details.]

B. No — A is not regular, as we now prove. Assume for the sake of contradiction that A is regular. Let p be the pumping length from the pumping lemma. Let $w = 01^p01^p$. Because $w \in A$ and $|w| \geq p$, the pumping lemma guarantees the existence of strings x, y, z such that $w = xyz$, $|xy| \leq p$, $|y| \geq 1$, and $xy^kz \in A$ for all $k \geq 0$. The first three properties imply that y is a non-empty substring of the first 01^p in w. Now consider the string $xy^0z = xz$. There are two cases.

1. If y contains 0, then xy^0z contains just one 0 and hence is not in A.
2. If y consists solely of 1s, then $xy^0z = 01^i01^j$ for some $i < j$, and hence is not in A.

In both cases we have shown that $xy^0z \notin A$. This fact contradicts the pumping lemma. We conclude that A is not regular.

C.A. One solution is $07\Sigma\Sigma\Sigma\Sigma\Sigma\Sigma\Sigma\Sigma\Sigma$. If you don't like Σ in regular expressions, then replace each Σ with $(0 \cup 1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)$.

C.B. [In reality, US mobile phone numbers have ten digits $a_0 \cdots a_9$ rather than nine. I counted them incorrectly while writing the exam. Fortunately, this error does not materially affect the problem.] To save writing, let R be the regular expression $(2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)$. Then a solution is

$$R\Sigma\Sigma R((0 \cup R)\Sigma \cup \Sigma(0 \cup R))\Sigma\Sigma\Sigma\Sigma.$$

C.C. Let K be the set of UK mobile phone numbers; it is regular by part A. Let S be the set of USA mobile phone numbers; it is regular by part B. Then $K \cup S$ is regular and

$$(K \cup S)^c = K^c \cap S^c = A$$

is also regular. Hence there must be a regular expression that matches A.

D.A. TRUE. [A finite language A is a union of one-string languages. For any one-string language, you can design a DFA or regular expression. Then A is regular because the class of regular languages is closed under finite unions.]

D.B. FALSE. [For example, let $A = \{0, 1\}^*$ and $B = \{0^n1^n : n \geq 0\}$.]

D.C. FALSE. [For example, working over $\Sigma = \{0, 1\}$, let $A = \{011\}$ and $B = \{1, 11\}$. Then $A/B = \{0, 01\}$ and $(A/B)B = \{01, 011, 0111\}$.]

D.D. FALSE. [For example, let $A_i = \{0^i1^i\}$. Then $A_0 \cup A_1 \cup A_2 \cup \cdots = \{0^n1^n : n \geq 0\}$.]

D.E. FALSE. [Here's the sketch of a proof. For any given alphabet Σ, there are infinitely many regular languages but only finitely many NFAs with three states. Namely, there are no more than

$$p \cdot 2^p \cdot (2^p)^{p(|\Sigma|+1)}$$

NFAs of p states. I asked this question because, in the part of the course about CFGs and PDAs, the analogous statement is essentially (but not exactly) true.]