A. Assume for the sake of contradiction that A is context-free. By the pumping lemma, there exists a pumping length p. Let $r = \mathtt{a}^p\mathtt{b}^p\mathtt{c}\mathtt{a}^p\mathtt{b}^p$. Then $r \in A$ and $|r| \geq p$, so $r = uvxyz$, where these strings satisfy certain properties. In particular, $|vxy| \leq p$ and $vy \neq \varepsilon$. There are three cases.

In the first case, vxy is a substring of the first $\mathtt{a}^p\mathtt{b}^p$ in r. Then pumping r down to $uv^0xy^0z = uxz$ removes some of the first $\mathtt{a}^p\mathtt{b}^p$ in r. So $uv^0xy^0z \notin A$.

In the second case, vxy is a substring of the second $\mathtt{a}^p\mathtt{b}^p$ in r. Then pumping r up to $uv^2xy^2z = uvvxyyz$ adds characters to the second $\mathtt{a}^p\mathtt{b}^p$ in r. So $uv^2xy^2z \notin A$.

In the third case, vxy is a substring of the middle $\mathtt{b}^p\mathtt{c}\mathtt{a}^p$ in r. There are three subcases. If vy contains any bs, then pumping down decreases the number of bs on the left side of r while maintaining the number of bs on the right side of r, so the pumped string is not in A. If vy contains any as, then pumping up similarly causes the string to leave A. If vy contains c, then pumping up or down causes the string to leave A.

In all three cases, we have shown that r can be pumped to leave A. This result contradicts the pumping lemma. This contradiction implies that A is not context-free.

B. Here is a Turing machine to decide A:

1. Using one left-to-right scan of the tape, wrap the input in turnstiles.
2. Using one left-to-right scan of the tape, check that the input is an element of

$$L((\mathtt{a} \cup \mathtt{b})^*\mathtt{c}(\mathtt{a} \cup \mathtt{b})^*).$$

   If not, then reject.

3. Repeatedly scan the tape left-to-right, until there are no unmarked characters after the c. On each scan:
   (a) Mark the first unmarked character before the c, remembering which character it is in state. (If there is no character to mark, then reject.)
   (b) Mark the first unmarked character after the c. If the two characters just marked are different, then reject.
4. Accept.

C.A. Assume for the sake of contradiction that B is decidable. Let H be a decider for B. Define a Turing machine D that, on input $\langle M \rangle$:

1. Modifies M to a Turing machine N that is identical to M, except that N has one more state, which is disconnected from the rest of its state diagram, and this new state is N's reject state. (Therefore N halts on an input exactly when M accepts the input.)

2. Runs H on $\langle N \rangle$ and outputs whatever H outputs.

This D is a decider, because both of its steps always halt. Moreover,

$$\begin{aligned} D \text{ accepts } \langle M \rangle &\Leftrightarrow H \text{ accepts } \langle N \rangle \\ &\Leftrightarrow N \text{ halts on input } \varepsilon \\ &\Leftrightarrow M \text{ accepts } \varepsilon \\ &\Leftrightarrow \varepsilon \in L(M). \end{aligned}$$

Therefore D is a decider for the language $\{\langle M \rangle : \varepsilon \in L(M)\}$. But this language is undecidable by Rice's theorem. This contradiction implies that B is undecidable.

C.B. Yes, B is recognizable. A recognizer R, on input $\langle M \rangle$, simply runs M on ε, accepting if and only if M halts.

C.C. No, B^c is not recognizable, for if both B and B^c were recognizable, then B would be decidable.