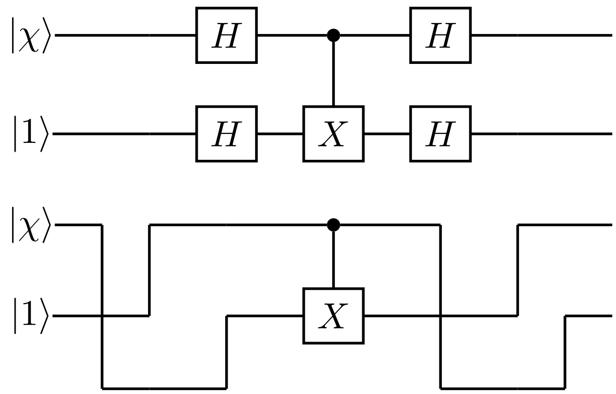

Figure 1: Two two-qbit circuits. The top circuit is $(H \otimes H) \cdot \text{CNOT} \cdot (H \otimes H)$. The bottom circuit is $\text{SWAP} \cdot \text{CNOT} \cdot \text{SWAP}$.

A. For one-qbit gates U and V, prove the identity $(U \otimes V)^* = U^* \otimes V^*$ algebraically. Also, explain why this identity makes sense on the level of circuit diagrams.

B. Figure 1 shows two circuits. Suppose that we input a two-qbit state of the form $|\chi\rangle \otimes |1\rangle$. Think about what these circuits do to the left qbit.

In the top circuit, the left qbit's state passes through H, then effectively I (because CNOT just passes along the left qbit's state), then H. So the left qbit undergoes $HIH = HH = I$ and is unchanged.

In the bottom circuit, the states swap, so $|1\rangle$ is the control state on CNOT, so $|\chi\rangle$ gets logically negated to $X\,|\chi\rangle$ before getting swapped back. So, overall, the circuit logically negates the left qbit.

So here's the problem for you: Compute the 4×4 matrices for the two circuits. (If you like, use our Python library. Feel free to do some rounding, such as $0.999999 + 0.000001i \approx 1$.) Then explain what's wrong with one or both of the stories above.

C. Suppose that a two-qbit state $|\psi\rangle$ enters a CNOT gate unentangled. Show that the state that emerges from the CNOT is entangled — except in rare special cases, which you should explicitly enumerate.

D. Find the 4×4 complex matrix A such that $A(|\alpha\rangle \otimes |\beta\rangle) = |\alpha\rangle \otimes |\alpha\rangle$ for all classical states $|\alpha\rangle, |\beta\rangle$. Then prove that $A(|\chi\rangle \otimes |\phi\rangle) \neq |\chi\rangle \otimes |\chi\rangle$ for some quantum states $|\chi\rangle, |\phi\rangle$. (The point of this exercise is that no-cloning is not a consequence of requiring *unitarity* of gates. It's a consequence of simply requiring *linearity* of gates.)

Finally, here are two optional questions. I can't put them on the official homework, because we haven't finished studying Deutsch yet. But I'm giving them to you now, in case you want to study them before the exam.

E (Optional). Consider Deutsch's algorithm to solve Deutsch's problem. What happens if, instead of feeding $|1\rangle\,|1\rangle$ into the circuit, we feed $|0\rangle\,|0\rangle$ into the circuit? Does the algorithm still solve the problem, or does it solve another problem, or does it do nothing of value?

F (Optional). Consider Deutsch's algorithm to solve Deutsch's problem. What if we replace the latter $H \otimes H$ with $H \otimes I$? Does the algorithm still solve the problem, or does it solve another problem, or does it do nothing of value?