A. Prove that, if $|\psi\rangle$ and $|\phi\rangle$ are unit vectors, then so is $|\psi\rangle \otimes |\phi\rangle$. (Therefore the tensor product of two states is a legitimate state.)

B. Using basic facts about how matrix products and tensor products interact, prove that, if U and V are unitary, then so is $U \otimes V$. (Therefore the tensor product of two gates is a legitimate gate.)

C. What is $H^{\otimes n} \cdot H^{\otimes n}$?

D. In the next chunk of our Python project, we will improve `first` and `last` so that they operate on n-qbit states. These functions are difficult to test, because they are probabilistic. So it's important that we have the math nailed down before we translate it into code.

With that motivation, here is your assignment: Let $|\psi\rangle$ be an n-qbit state. Give a precise mathematical definition of partial measurement of the last qbit. Also, explain how you would compute all of the quantities involved in terms of the entries of $|\psi\rangle$.

E. Let U be any n-qbit gate. Define a function $\mathrm{CU} : \mathbb{C}^{2^{n+1}} \to \mathbb{C}^{2^{n+1}}$ by declaring, for any n-qbit state $|\psi\rangle$, that

$$\mathrm{CU}(|0\rangle \otimes |\psi\rangle) = |0\rangle \otimes |\psi\rangle \text{ and } \mathrm{CU}(|1\rangle \otimes |\psi\rangle) = |1\rangle \otimes U \cdot |\psi\rangle .$$

Find the $2^{n+1} \times 2^{n+1}$ matrix that represents CU, and prove that that matrix is unitary. (Here's one way to check your work: CNOT is the special case where $U = X$.)

F. For any n-qbit state $|\rho\rangle$, define a $2^n \times 2^n$ matrix R by $R = 2\,|\rho\rangle\langle\rho| - I$. Verify that R acts on $\mathbb{C}^{2^n}$ as reflection across $|\rho\rangle$, by completing the following problems.

1. Prove that $R\,|\rho\rangle = |\rho\rangle$.
2. Prove that if $|\psi\rangle$ is perpendicular to $|\rho\rangle$, then $R\,|\psi\rangle = -\,|\psi\rangle$.
3. Prove that $R^2 = I$.
4. Also prove that R is unitary.