All of these problems are optional. They are intended to help you study for Exam C, so I urge you to solve them before Friday. But do not bother to write/type serious solutions, because you will not submit them for grading.

A. Prove that, if S is a skew-Hermitian $m \times m$ matrix, then iS is Hermitian. (This task is short. You might have done it in class. We definitely did something similar.)

B. Prove that $\frac{d}{dt} e^{tA} = A e^{tA}$. (This task is short. You have to differentiate a power series term-by-term. If this is daunting, then don't worry very much. I won't ask you to do tasks like this on the exam. I'm putting this problem here mainly because the next problem relies on it.)

C. Our continuous-time concept of quantum computation relies either on unitary evolution

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle ,$$

where $U(t)$ is unitary, or on Schrödinger's equation

$$i \frac{d}{dt} |\psi(t)\rangle = Q(t) |\psi(t)\rangle ,$$

where $Q(t)$ is Hermitian. In class we showed that the latter follows from the former. This problem partially shows that the former follows from the latter, while connecting this continuous-time treatment back to the discrete-time gate model, which we used for most of the course.

1. Assume that Q is constant with respect to time. Prove that $|\psi(t)\rangle = e^{-itQ} |\psi(0)\rangle$ satisfies the Schrödinger equation, no matter what the initial state $|\psi(0)\rangle$ is. (This task is short. And then, by the theory of ordinary differential equations, it follows that this $|\psi(t)\rangle$ is the *unique* solution to the Schrödinger equation with that Q and that initial condition.)

2. Juanita, who is an experimental physicist, is graciously operating our continuous-time quantum computer for us. She imposes a constant Q between $t = 0$ and $t = 1$. Show that

   $$|\psi(1)\rangle = U |\psi(0)\rangle ,$$

   for some unitary matrix U. We think of U as a gate that acts when a clock ticks. Express how U depends on Q explicitly. (This task is short.)

D. This problem is inspired by the adiabatic treatment of the simplest case of the satisfiability problem. Let $\tilde{Q}^{(0)} = \frac{1}{2}(I - X)$, $\tilde{Q}^{(1)} = \frac{1}{2}(I + Z)$, and

$$\tilde{Q}(\tilde{t}) = (1 - \tilde{t}) Q^{(0)} + \tilde{t} Q^{(1)}.$$

Notice that $\tilde{Q}(\tilde{t})$ interpolates between $\tilde{Q}(0) = \tilde{Q}^{(0)}$ and $\tilde{Q}(1) = \tilde{Q}^{(1)}$ as $\tilde{t}$ varies between 0 and 1.

1. Prove that $\tilde{Q}(\tilde{t})$ is Hermitian. (This task is medium-short.)

2. Show that the eigenvalues of $\tilde{Q}(\tilde{t})$ are

$$\frac{1 \pm \sqrt{1 - 2\tilde{t} + 2\tilde{t}^2}}{2}.$$

   (This task is not short. On a timed, in-class exam, I would not ask you to compute complicated eigenvalues or eigenvectors like this.)

3. For $\tilde{t} \in [0, 1]$, what's the minimum gap between the least and second-least eigenvalues of $\tilde{Q}(\tilde{t})$? (This task is medium-short. It's important because the minimum gap governs how slowly Juanita has to run the apparatus.)