A. div $\langle 2x+3y-z, y+2z, -x-3z\rangle = 2+1+-3 = 0$.

B. Let $\vec{F} = \langle 6xy+2, 3x^2+6y\rangle$. First we check that $\partial F_2/\partial x = 6x = \partial F_1/\partial y$. Good. Now, if $\vec{F} = \nabla f$, then

$$\begin{aligned} f &= \int F_1 \, \partial x \\ &= \int 6xy + 2 \, \partial x \\ &= 3x^2y + 2x + g(y), \end{aligned}$$

where $g(y)$ is some unknown function. Then

$$\begin{aligned} 3x^2 + 6y &= F_2 \\ &= \partial f/\partial y \\ &= \frac{\partial}{\partial y}(3x^2y + 2x + g(y)) \\ &= 3x^2 + g'(y). \end{aligned}$$

So we find that $g'(y) = 6y$ and $g(y) = 3y^2 + C$. Putting it all together, we have

$$f(x,y) = 3x^2y + 2x + 3y^2 + C.$$

C. [Draw a picture.] The curve C is the counterclockwise boundary of a certain region D, so we can use Green's theorem:

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{x} &= \iint_D \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x} \, dx \, dy \\ &= \int_0^{\sqrt{3}} \int_{-1+y/\sqrt{3}}^{1-y/\sqrt{3}} y \, dx \, dy \\ &= \int_0^{\sqrt{3}} 2y(1 - y/\sqrt{3}) \, dy \\ &= \left[y^2 - \frac{2}{3\sqrt{3}} y^3 \right]_0^{\sqrt{3}} \\ &= \left(3 - \frac{2}{3\sqrt{3}} 3\sqrt{3} \right) - (0 - 0) \\ &= 1. \end{aligned}$$

[Some students successfully computed three line integrals, got the same answer, and earned full credit. But that approach is not usually successful.]

D. Work is the line integral of the force field along the trajectory C:

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{x} &= \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t) \, dt \\ &= \int_a^b (k\vec{x}\,'(t) \times \vec{B}(\vec{x}(t))) \cdot \vec{x}\,'(t) \, dt. \end{aligned}$$

But notice that $\vec{x}\,'(t)$ is perpendicular to $\vec{x}\,'(t) \times \vec{B}(\vec{x}(t))$. Therefore the integrand is 0 and the integral is 0.

[It is not important that the trajectory is a straight line. The work is 0 along any trajectory. So, in general, how do magnetic fields do work on particles? My understanding is that they don't do work; however, they are almost always accompanied by electric fields, which do the work for them.]

E. [This problem was among the review problems that I posted in the days leading up to the exam — except that I rotated it $\pi/2$ about the z-axis, to make it slightly different. Anyway, draw a picture.] The simplest answer is the $dx\ dy\ dz$ integral

$$\int_0^2 \int_{-z}^{2-2z} \int_{y/2+z-1}^{-y/2-z+1} f(x,y,z)\ dx\ dy\ dz.$$

[The three next simplest orders of integration are $dx\ dz\ dy$, $dy\ dx\ dz$, and $dy\ dz\ dx$:

$$\begin{aligned}
&\int_{-2}^0 \int_{-y}^{1-y/2} \int_{y/2+z-1}^{-y/2-z+1} f(x,y,z)\ dx\ dz\ dy \\
&+ \int_0^2 \int_0^{1-y/2} \int_{y/2+z-1}^{-y/2-z+1} f(x,y,z)\ dx\ dz\ dy \\
= &\int_0^2 \int_{z/2-1}^0 \int_{-z}^{2x-2z+2} f(x,y,z)\ dy\ dx\ dz \\
&+ \int_0^2 \int_0^{1-z/2} \int_{-z}^{-2x-2z+2} f(x,y,z)\ dy\ dx\ dz \\
= &\int_{-1}^0 \int_0^{2+2x} \int_{-z}^{2x-2z+2} f(x,y,z)\ dy\ dz\ dx \\
&+ \int_0^1 \int_0^{2-2x} \int_{-z}^{-2x-2z+2} f(x,y,z)\ dy\ dz\ dx.
\end{aligned}$$

The two most complicated orders of integration are $dz\ dx\ dy$ and $dz\ dy\ dx$:

$$\begin{aligned}
&\int_{-2}^{0}\int_{-1-y/2}^{0}\int_{-y}^{x-y/2+1} f(x,y,z)\ dz\ dx\ dy \\
&+\int_{-2}^{0}\int_{0}^{1+y/2}\int_{-y}^{-x-y/2+1} f(x,y,z)\ dz\ dx\ dy \\
&+\int_{0}^{2}\int_{-1+y/2}^{0}\int_{0}^{x-y/2+1} f(x,y,z)\ dz\ dx\ dy \\
&+\int_{0}^{2}\int_{0}^{1-y/2}\int_{0}^{-x-y/2+1} f(x,y,z)\ dz\ dx\ dy \\
=\ &\int_{-1}^{0}\int_{-2x-2}^{0}\int_{-y}^{x-y/2+1} f(x,y,z)\ dz\ dy\ dx \\
&+\int_{0}^{1}\int_{2x-2}^{0}\int_{-y}^{-x-y/2+1} f(x,y,z)\ dz\ dy\ dx \\
&+\int_{-1}^{0}\int_{0}^{2+2x}\int_{0}^{x-y/2+1} f(x,y,z)\ dz\ dy\ dx \\
&+\int_{0}^{1}\int_{0}^{2-2x}\int_{0}^{-x-y/2+1} f(x,y,z)\ dz\ dy\ dx.
\end{aligned}$$

I have checked all six answers in Mathematica — for example, they all evaluate to $4/3$ when $f=1$ — but they might contain typographical errors from transcription.]

F. Roughly speaking, the product rule should look like

$$(f\vec{F})' = f'\vec{F} + f\vec{F}'.$$

But we need to figure out which kinds of derivatives we should be using (options include gradient, curl, and divergence) and which kinds of multiplications we should be using (options include scalar multiplication, dot product, and cross product). A crucial insight is that, because the curl is a vector field, both terms on the right must be vector-valued. So I guess

$$\operatorname{curl}(f\vec{F}) = (\nabla f)\times\vec{F} + f\operatorname{curl}\vec{F}.$$

[By the way, this product rule is indeed correct. Verify it algebraically, if you like.]

G. [We did this entire thing in class early in the course. Draw a picture with a plane passing through the point $\vec{p}$ perpendicular to the vector $\vec{n}$ based at $\vec{p}$. Also label an arbitrary point $\vec{x}$ on the plane. Also draw the vector $\vec{x}-\vec{p}$ going from $\vec{p}$ to $\vec{x}$. Optionally, draw the origin $\vec{0}$ somewhere and draw the vectors $\vec{p}$ and $\vec{x}$ from the origin to those points respectively.] A point $\vec{x}$ is on the plane if and only if $\vec{x}-\vec{p}$ is perpendicular to $\vec{n}$. So $\vec{x}$ is on the plane if and only if

$$0 = \vec{n}\cdot(\vec{x}-\vec{p}) = \vec{n}\cdot\vec{x} - \vec{n}\cdot\vec{p},$$

which holds if and only if $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$.

H.A. Let x be the number of screens made and y the number of phones made. Then the profit is

$$f(x, y) = 300y^{1/2} + 80(x - y)^{1/2}.$$

The constraints are $y \geq 0$, $x \geq y$, and $x \leq 10^6$. So we are trying to maximize $f(x, y)$ over a triangle with vertices $(0, 0)$, $(10^6, 0)$, $(10^6, 10^6)$. [I would draw a picture of this triangle.]

[Some students instead said: Let s be the extra screens made and p the phones made. Then the profit is $f(s, p) = 80s^{1/2} + 300p^{1/2}$. The domain of optimization is defined by $s \geq 0$, $p \geq 0$, and $s + p \leq 10^6$. Again I would draw a picture of that triangle and proceed much as in H.B below.]

[Some students made a further leap, in assuming that obviously the entire supply of rare-earth materials should be used. So they proposed that we optimize $f(s, p)$ subject to $g(s, p) = s + p = 10^6$, with the additional requirements that $s \geq 0$ and $p \geq 0$. Then the problem seems treatable by Lagrange multipliers, although some of the details are strange. Some students used the constraint to solve for p in terms of s (or vice-versa), thus reducing the whole problem to a single-variable problem solvable by the methods of Calculus 1. In the future, I will try harder to make that impossible.]

H.B. [Many students framed a problem that was not treatable by Lagrange multipliers, and then proceeded by Lagrange multipliers anyway, losing many points.] We are maximizing a continuous function on a closed, bounded region. So we know that f achieves a maximum, and that the maximum must occur at a critical point of f or at a boundary point of the region. To find the critical points, we compute

$$\nabla f = \langle \partial f/\partial x, \partial f/\partial y \rangle = \langle 40(x - y)^{-1/2}, 150y^{-1/2} - 40(x - y)^{-1/2} \rangle.$$

The gradient is never $\vec{0}$. The gradient is undefined where $x = y$ and where $y = 0$. These points occur on the boundary of the region. So the boundary points are the only points of interest.

In general optimization problems of this kind, we must study all boundary points. However, in this problem f increases with x, so the maximum must occur somewhere on the right side of the boundary, where $x = 10^6$ and $0 \leq y \leq 10^6$.

So the problem boils down to maximizing $g(y) = 300y^{1/2} + 80(10^6 - y)^{1/2}$ over the interval $[0, 10^6]$. This is a Calculus I problem, so I won't show all of the details. It turns out that $g'(y)$ is never 0, and it's undefined only at $y = 0$ and $y = 10^6$, which are the boundary points of the interval. The profits at those two points are respectively

$$g(0) = f(10^6, 0) = 80(10^6)^{1/2} = 80{,}000$$

and

$$g(10^6) = f(10^6, 10^6) = 300(10^6)^{1/2} = 300{,}000.$$

So the company should make as many screens as it can ($x = 10^6$) and use all of them in its own phones ($y = 10^6$), for a profit of \$300,000.

[This result makes intuitive sense. The company should use all of its rare-earth material supply, and it should prefer high-profit phones over lower-profit screens. However, \$300,000 isn't much profit on 1,000,000 phones, right? The issue is the square roots in the profit function. The square roots are intended to capture an effect, in which rare gadgets command premium prices from enthusiasts but commodity gadgets barely make any money at all. To some extent this dynamic plays out in the actual phone market, but the square roots are unrealistically pessimistic, it seems.]