A.A. The desired vector is $\vec{w} - \vec{v} = \langle -6, 0, 3 \rangle$.

A.B. The projection is

$$\text{proj}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{-8 + 1 - 2}{16 + 1 + 4} \langle -4, 1, 2 \rangle = \frac{-3}{7} \langle -4, 1, 2 \rangle = \left\langle \frac{12}{7}, \frac{-3}{7}, \frac{-6}{7} \right\rangle.$$

A.C. The desired vector is

$$3 \frac{\vec{v}}{|\vec{v}|} = \frac{3}{\sqrt{6}} \langle 2, 1, -1 \rangle = \left\langle \sqrt{6}, \frac{3}{\sqrt{6}}, \frac{-3}{\sqrt{6}} \right\rangle.$$

B. No, it need not be true that $\vec{v} = \vec{u}$. For example, reducing the angle between $\vec{v}$ and $\vec{w}$ can be compensated by reducing the length of $\vec{u}$. An explicit example is

$$\langle 0, 1, 0 \rangle \times \langle 0, 0, 1 \rangle = \langle 1, 0, 0 \rangle = \langle 0, 1, 1 \rangle \times \langle 0, 0, 1 \rangle.$$

An even simpler example, which a student devised, is $\vec{u} = \vec{0}$ and $\vec{v} = \vec{w} \neq \vec{0}$.

C.A. Using L'Hopital's rule once, we have

$$\lim_{t \to 0} \vec{x}(t) = \left(\lim_{t \to 0} 1 + t^2, \lim_{t \to 0} \frac{\sin t}{t} \right) = \left(1, \lim_{t \to 0} \frac{\cos t}{1} \right) = (1, 1).$$

C.B. No, $\vec{x}$ is not continuous at $t = 0$, because it is not even defined there.

C.C. By the quotient rule,

$$\vec{x}\,'(t) = \left(\frac{d}{dt} \left(1 + t^2\right), \frac{d}{dt} \left(\frac{\sin t}{t} \right) \right) = \left(2t, \frac{t \cos t - \sin t}{t^2} \right).$$

C.D. Anti-differentiation by substitution gives

$$\int \vec{y}(t)\ dt = \left(\int t^3\ dt, \int \frac{2t}{\sqrt{t^2 + 1}}\ dt \right) = \left(\frac{1}{4} t^4 + C_1, 2(t^2 + 1)^{1/2} + C_2 \right).$$

[It is important that each component anti-derivative have a "+ C". It is important that the two constants are allowed to be different.]

D.A. Following our usual method, we write

$$y = x^2 \quad \Rightarrow \quad r \sin \theta = r^2 \cos^2 \theta \quad \Rightarrow \quad r = \frac{\sin \theta}{\cos^2 \theta}.$$

D.B. [It helps to draw a picture.] We need θ to go from 0, where $r = 0$, to $\pi/2$, where r is undefined. Then we need θ to go from $\pi/2$ to π, where $r = 0$ again. So the answer is $[0, \pi/2) \cup (\pi/2, \pi]$. It would also be fine to leave off one of those endpoints.

E.A. [Instead of drawing, I'll describe my thought process.] First, $(\cos t, \sin t)$ plots as the unit circle. So $(\cos t, 2 \sin t)$ plots as the ellipse of width 2 and height 4. Then $(1 + \cos t, 2 \sin t)$ plots as that ellipse shifted right by 1, so that it is centered at $(1, 0)$.

E.B. The arc length is

$$\int_0^{2\pi} |\vec{x}\,'(t)| \, dt = \int_0^{2\pi} |\langle -\sin t, 2 \cos t \rangle| \, dt = \int_0^{2\pi} \sqrt{\sin^2 t + 4 \cos^2 t} \, dt.$$

We can do a little more algebra on the integrand, but it doesn't really help. We can't compute the integral symbolically. [My intent, in writing this problem, was just to see whether students could set up the arc length integral. See the nearly identical example in the Mathematica notebook `calculusCurves.nb` on our course web site.]

F. [Draw a picture.] A normal vector to the hyperplane is $\vec{n} = \langle a, b, c, d \rangle$. Let $\vec{x}$ be any point on the hyperplane. Then the equation of the hyperplane says that $\vec{n} \cdot \vec{x} = e$. The distance from $\vec{p}$ to the hyperplane is the length of the projection of the vector $\vec{p} - \vec{x}$ onto $\vec{n}$:

$$\begin{aligned}
|\mathrm{proj}_{\vec{n}}(\vec{p} - \vec{x})| &= \left| \frac{\vec{n} \cdot (\vec{p} - \vec{x})}{\vec{n} \cdot \vec{n}} \vec{n} \right| \\
&= \left| \frac{\vec{n} \cdot \vec{p} - \vec{n} \cdot \vec{x}}{|\vec{n}|} \frac{\vec{n}}{|\vec{n}|} \right| \\
&= \frac{|\vec{n} \cdot \vec{p} - \vec{n} \cdot \vec{x}|}{|\vec{n}|} \\
&= \frac{|ap_1 + bp_2 + cp_3 + dp_4 - e|}{\sqrt{a^2 + b^2 + c^2 + d^2}}.
\end{aligned}$$