A. No, the limit in question does not exist. If we approach along the line $x = 0$, then the function is identically 0, so the limit is 0. If we approach along the line $y = 0$, then the function takes the form x^{-2}, which goes to ∞ at the origin. For the limit to exist, it must exist and have the same value from every approach direction. That's not true of this function, so the limit does not exist.

B. First, we're going to need ∇V. We compute

$$\begin{aligned}\frac{\partial V}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(-GMm(\vec{x}\cdot\vec{x})^{-1/2}\right)\\ &= \frac{1}{2}GMm(\vec{x}\cdot\vec{x})^{-3/2}2x_1\\ &= \frac{GMm}{|\vec{x}|^3}x_1.\end{aligned}$$

And similarly we get

$$\frac{\partial V}{\partial x_2} = \frac{GMm}{|\vec{x}|^3}x_2, \quad \frac{\partial V}{\partial x_3} = \frac{GMm}{|\vec{x}|^3}x_3.$$

Therefore

$$\nabla V = \frac{GMm}{|\vec{x}|^3}\vec{x}.$$

Second, the question is asking for the directional derivative of V at $\vec{x}$ in the direction $-\vec{x}/|\vec{x}|$, which is

$$D_{-\vec{x}/|\vec{x}|}V = \nabla V\cdot\left(-\frac{\vec{x}}{|\vec{x}|}\right) = -\frac{GMm}{|\vec{x}|^4}\vec{x}\cdot\vec{x} = -\frac{GMm}{|\vec{x}|^2}.$$

[An alternative way to do the second part is to parametrize the line through $\vec{x}$ moving toward the origin with speed 1. To make this clearer, name the point in question $\vec{x}_0$. Then the parametrization is $\vec{x}(t) = \vec{x}_0 + t(-\vec{x}_0/|\vec{x}_0|)$. Then we want dV/dt, which by the chain rule equals $\nabla V\cdot\vec{x}\,' = \nabla V\cdot(-\vec{x}/|\vec{x}|)$, which leads to the same answer.]

C.A. By the chain rule, $\frac{d}{dt}y = \nabla y\cdot\vec{x}\,'$. Here, it is understood that ∇y can be written as a function of t by plugging in $\vec{x}(t)$ for $\vec{x}$.

C.B. By part A and the product rule,

$$\frac{d}{dt}\frac{d}{dt}y = \frac{d}{dt}\left(\nabla y\cdot\vec{x}\,'\right) = \left(\frac{d}{dt}\nabla y\right)\cdot\vec{x}\,' + \nabla y\cdot\vec{x}\,''.$$

C.C. In this example,

$$\begin{aligned}\nabla y &= \langle x_2, x_1\rangle\\ &= \langle \sin t, t^2\rangle\\ \Rightarrow \quad \frac{d}{dt}\nabla y &= \langle \cos t, 2t\rangle.\end{aligned}$$

Meanwhile, $\vec{x}\,' = \langle 2t, \cos t \rangle$ and $\vec{x}\,'' = \langle 2, -\sin t \rangle$. Thus our answer from part B is

$$\begin{aligned} \frac{d}{dt}\frac{d}{dt}y &= \left(\frac{d}{dt}\nabla y \right) \cdot \vec{x}\,' + \nabla y \cdot \vec{x}\,'' \\ &= \langle \cos t, 2t \rangle \cdot \langle 2t, \cos t \rangle + \langle \sin t, t^2 \rangle \cdot \langle 2, -\sin t \rangle \\ &= 4t \cos t + (2 - t^2) \sin t. \end{aligned}$$

To check this answer, we can express the original y as a function of t and differentiate twice:

$$\begin{aligned} \frac{d}{dt}\frac{d}{dt}y &= \frac{d}{dt}\frac{d}{dt}\left(t^2 \sin t\right) \\ &= \frac{d}{dt}\left(2t \sin t + t^2 \cos t\right) \\ &= 2 \sin t + 2t \cos t + 2t \cos t - t^2 \sin t. \end{aligned}$$

Yes, the answers agree.

D.A. We want to maximize the profit function

$$p(x, y) = r(x, y) - c(x, y) = 6 - \frac{1}{16}x^3 - \frac{1}{8}x^2 + x - \frac{1}{2}y^2 + y$$

on the closed disk D consisting of points (x, y) such that $x^2 + y^2 \leq 4$.

D.B. Because we are optimizing a continuous function f on a closed, bounded region D, we know that the maximum must occur either among the critical points of f or along the boundary of D. We compute

$$\nabla f = \left\langle -\frac{3}{16}x^2 - \frac{1}{4}x + 1, -y + 1 \right\rangle.$$

This gradient is never undefined. So, to find the critical points, we must solve the system of equations

$$\begin{aligned} -\frac{3}{16}x^2 - \frac{1}{4}x + 1 &= 0, \\ -y + 1 &= 0. \end{aligned}$$

Every point in D satisfying those equations must be checked. Also, every point where $x^2 + y^2 = 4$ must be checked.

D.C. The second $\nabla f = \vec{0}$ equation implies that $y = 1$. The first equation implies that

$$x = \frac{(1/4) \pm \sqrt{1/16 - -12/16}}{-3/8} = \frac{(1/4) \pm (1/4)\sqrt{13}}{-3/8} = -\frac{2}{3}\left(1 \pm \sqrt{13}\right).$$

Are these points in the disk D? Because $\sqrt{13} > 3$, we can estimate that $1 + \sqrt{13} > 4$ and $-\frac{2}{3}(1 + \sqrt{13}) < -8/3 < -2$. So the critical point $\left(-\frac{2}{3}(1 + \sqrt{13}), 1\right)$ is outside D. The other critical point, $\left(-\frac{2}{3}(1-\sqrt{13}), 1\right)$, is harder to test without a calculator, so let's keep it in our list of points to check.

We must also check the boundary of D, where $x^2 + y^2 = 4$. One approach is to parametrize the boundary. That is, plug $x = 2\cos t$ and $y = 2\sin t$ into p and solve the resulting Calculus 1 problem. Another way is Lagrange multipliers along the boundary. A third, shorter way is to notice that the maximum must occur where $y = 1$, because the revenue is maximized where $y = 1$ and the cost doesn't depend on y at all. Following this third way, we see that there are only two important boundary points: $(\pm\sqrt{3}, 1)$.

So we have three points to check: $\left(-\frac{2}{3}(1-\sqrt{13}), 1\right)$, $(-\sqrt{3}, 1)$, and $(\sqrt{3}, 1)$. Without a calculator this checking is a lot of tedious work unrelated to Calculus 3, so I stop here.

[Using Mathematica later, outside the context of the exam, here's what I found. The critical point $\left(-\frac{2}{3}(1-\sqrt{13}), 1\right)$ is just outside D, so it cannot be the solution. The boundary point $(-\sqrt{3}, 1)$ produces $p \approx 4.7$. The boundary point $(\sqrt{3}, 1)$ produces $p \approx 7.5$. So we should place the store at $(\sqrt{3}, 1)$ and make profit approximately 7.5.]

[Another approach is to plug $y = 1$ into p, turning the profit into a function of x alone. Then the profit $f(x) = p(x, 1)$ can be maximized on $[-\sqrt{3}, \sqrt{3}]$ using Calculus 1. One can show that $f'(x) > 0$ on $[-\sqrt{3}, \sqrt{3}]$, so that the maximum occurs at $x = \sqrt{3}$ as above. However, a solution relying entirely on this approach does not earn full credit, because it contradicts the instruction to formulate the profit maximization problem using techniques of this course.]