1 Arctic

Suppose that we want to model how the area A of open sea water in the Arctic varies over time t. We can take inspiration from the University of Minnesota M.S. thesis "A Simple Mathematical Model of Arctic Ocean Ice Extent" by Wen Xing. After being simplified a bit for the sake of Math 211, the model says that

$$A = ce^{kFt},$$

where $c, k > 0$ are constants and F is the atmospheric CO_2 level. This F can be further modeled as $F = at + b$, where $a, b > 0$ are constant.

A. Draw a diagram showing the variables A, F, and t, and how they depend on each other. Label each edge with its corresponding derivative, and compute that derivative. Compute dA/dt based on the diagram.

B. Also compute dA/dt by expressing A as a function of t alone and differentiating from there. Do you get the same answer?

Epilogue: Wen Xing then uses a statistical technique called regression to find values for a, b, c, k based on climate data. The model fits the data fairly well. The fact that A is exponential in t is worrisome.

2 Spacecraft

In these problems, we model how a spacecraft moves under the influence of Earth's gravity. Imagine Earth sitting at the origin in $\mathbb{R}^3$. A spacecraft is floating through space near Earth with trajectory $\vec{x}(t)$. The gravitational *potential energy* of the spacecraft relative to Earth is

$$V(\vec{x}) = -\frac{GMm}{|\vec{x}\,|},$$

where $M > 0$ is the constant mass of Earth, $m > 0$ is the constant mass of the spacecraft, and $G > 0$ is another constant. Roughly speaking, V is the answer to the question, "How much does the Earth want to crash the spaceship?" So it is reasonable to ask: How does the value of V at the spacecraft change over time?

A. Draw a diagram showing the variables V, x_1, x_2, x_3, and t, and how they depend on each other. Label each edge with its corresponding derivative. Compute dV/dt.

We haven't studied the gradient yet, but it's easy to define right now: The gradient ∇V is simply the vector of partial derivatives

$$\nabla V = \left\langle \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3} \right\rangle.$$

You can check, if you like, that the gravitational force $\vec{F}$ on the spacecraft is related to the gravitational potential energy V by $\vec{F} = -\nabla V$. We can also define the *kinetic energy* of the spacecraft as $K = mv^2/2$, where v is the spacecraft's speed. Assume that the spacecraft is not propelling itself right now; it's merely moving however gravity wants to move it. Then the total energy of the system is $K + V$, and this next exercise shows that it is *conserved* (meaning constant with respect to time).

B. Prove that the derivative of $K + V$ with respect to t is 0. You will need to use Newton's second law of motion: $\vec{F} = m\vec{a}$.

3 Company

A hypothetical computer company makes one model of laptop computer and one model of mobile phone. Within a single fiscal quarter (meaning a three-month period), the total cost c incurred by the company is

$$c = L\ell + Pp + a$$

dollars, where ℓ is the number of laptops made, p is the number of phones made, and a is the total spending on advertising.

A. What is the meaning of the constants L and P?

B. Draw a diagram showing all of the variables and how they depend on each other (based on the information given above). Label each edge with the corresponding derivative, and give the value of that derivative — or, if you don't know its value, then guess whether it's positive, negative, or zero.

Now we start incorporating more features into the model. Suppose that ℓ and p are chosen to match consumer demand. Well, consumer demand is influenced by advertising. So ℓ and p should be functions of a.

C. Repeat problem B with this new information. Also compute a derivative that expresses how c changes with respect to changes in a.

The company's phones are much more popular than its laptops. A consumer who buys the phone is not necessarily inclined to buy the laptop, but a consumer who buys the laptop often buys the phone too. Consequently the number of phones made is influenced by the number of laptops made.

D. Repeat problem C with this new information.

Epilogue: Let's stop there, even though there's much more to do. I've based these problems loosely on Apple, Inc., but Apple has many products, in many markets, made with many components from many suppliers, with many kinds of advertising and other costs. A realistic treatment of Apple's costs would probably involve thousands of variables, and there would be complicated, non-linear interactions among those variables. For example, as the number of batteries purchased increases, the cost per battery might decrease (because of volume discounts) until it started increasing (because of scarcity of raw materials). And we would want to compute not just the costs (money spent) but also the revenue (money earned). And we might want to optimize the difference between the two.