1 Filling a Water Tower or Reservoir

Suppose that a particle of mass m is sitting on the plane x-y-plane, where $z = 0$. (In this problem, mass is measured in kg, distance in m, and time in s.) To raise this particle against Earth's gravitational field to a given altitude $z > 0$ takes energy or *work*. The amount of work is mgz, where $g = 9.8$ m/s^2.

Now consider a solid, liquid, or gaseous body that occupies a region E of space and has density $\delta(x, y, z)$ (in kg/m^3). Assume that each particle in the body began in the $z = 0$ plane. The total work, to raise all of the particles in the body, from $z = 0$ to their current positions, equals $\iiint_E \delta g z \, dV$.

This concept can be used to calculate the energy needed to fill a water tower with water. Suppose that you have a cylindrical tank of radius R and height B. The tank is elevated, so that its bottom is at height A above the ground. The density of water is 1,000 kg/m^3.

A. Calculate the work needed to fill this water tower from the ground.

This amount of energy is important to local utilities, who literally pay these energy costs. It's also important for some renewable energy projects. For solar power, how do we store excess energy from sunny days, so that it can be released during nights and cloudy days? For wind power, how do we store excess energy from windy periods, so that it can released during non-windy periods? Well, one solution is to use excess energy to pump water up — into towers or reservoirs — then recover the energy by letting the water drain onto dynamos.

As a consultant in the energy industry, you're considering damming a river to form a reservoir that can be used for energy storage. You place coordinates so that the base of the proposed dam is at the origin, the x-axis runs upstream, and the z-axis is vertical. If your dam has height h, and the river valley upstream from the dam fills with water, then the water forms a tetrahedron with vertices $(0, 0, 0)$, $(0, 2h, h)$, $(0, -2h, h)$, and $(40h, 0, h)$.

B. If you release all of that water onto perfectly efficient dynamos at the base of the dam, then how much energy do you recover?

A more realistic version of this problem would be more difficult. How do you handle a valley that is not perfectly straight with perfectly straight slopes? How much water leaks or evaporates? What if the dynamos' efficiency depends on how quickly the water hits them?

(And what effect would this project have on wildlife? What about the people living in the village that would be flooded? Are there any archaeological sites in the valley? Would the reservoir mitigate or exacerbate water supply issues? It seems to me that the entire math problem is just one relatively simple step in a much bigger problem.)

2 Making Hawaii

Geologists use similar concepts to understand how energy is distributed among various geologic processes. For example, along the San Andreas fault in California, the tectonic plates underlying North America and the Pacific Ocean grind against each other. This grinding is very slow, but the plates are extremely massive, so that the kinetic energy of the system is gigantic. Any energy that goes into permanent topographic change (mountain building) is energy that's not available for earthquakes. So these energy considerations affect real people.

In that spirit, let's talk about the island of Hawaii. Essentially, it's one enormous mountain that extends down to the sea floor. See the illustration below, which is approximately to scale. It's a cone of height of 9,300 m and radius 124,000 m. Its density is a constant 3,000 kg/m^3.

C. How much work did it take to raise Hawaii from the sea floor?

Again, the foregoing problem is a simplification of reality. Hawaii is not really a cone of constant density. The acceleration of Earth's gravitational field is not a constant 9.8 m/s^2 over regions this large. In fact, Hawaii's mass noticeably alters the gravitational field around Hawaii. Geology is complicated.

3 Where is the Electron?

In quantum physics, a particle is described by a *wave function* ψ. The wave function for the 1s state of an electron in a hydrogen atom is $\psi(\rho) = (\pi a^3)^{-1/2} e^{-\rho/a}$, where ρ is the distance to the nucleus, and $a = 5.3 \cdot 10^{-11}$ m is a constant called the *Bohr radius*. The probability of finding the electron in a region E of space is $\iiint_E |\psi|^2 \, dV$.

D.A. Compute the probability of finding the electron at a distance of R or less from the nucleus.

D.B. What is the limit, as $R \to \infty$, of your answer from part A of this problem? Calculate the limit and explain why it makes sense.

4 Mass of the Atmosphere

In this problem, let's assume that the Earth is a spherical ball of radius $R = 6{,}371$ km. Measurements indicate that the density of the atmosphere drops off exponentially with altitude. To be precise, at an altitude of h km above the Earth's surface, the atmosphere has density $\delta(h) = ae^{-bh}$ kg/km^3, where $a = 1.225 \cdot 10^9$ and $b = 0.13$. There is no clear boundary between the atmosphere and space; the atmosphere just keeps getting thinner and thinner.

E. Calculate the mass of Earth's atmosphere. (Be careful in how you handle h.)