This exam should have nine pages, including this cover page and problems A–F spread out over eight pages.

You have 150 minutes to complete this exam.

No notes, books, calculators, computers, etc. are allowed, except for your two-sided, one-sheet crib sheet that we discussed.

For a function to be "smooth" means that derivatives of all orders exist, wherever that function is defined. In other words, the function in question is infinitely differentiable on its domain. For a parametrized curve, "smooth" further means that the velocity is never $\vec{0}$.

Show all of your work, in as organized a manner as possible. Incorrect answers with solid work often earn partial credit. Correct answers without explanatory work rarely earn full credit.

Pictures often help both you and your reader!

Perform as much algebraic simplification as you can. Simple correct answers are generally preferred over complicated correct answers. Do simple arithmetic, but don't bother to do complicated arithmetic. Mark your final answer clearly.

Good luck. :)

A. For smooth 3D vector fields $\vec{F}$ and $\vec{G}$ on $\mathbb{R}^3$, with dot product $\cdot$ and cross product $\times$, here are three guesses for a product rule for $\text{div}(\vec{F} \times \vec{G})$. For each one, show that it is correct or incorrect. (It is conceivable that there are multiple correct guesses or no correct guesses, so evaluate each part on its own. Show all work; don't simply cite from memory or crib sheet.)

A.A. $\text{div}(\vec{F} \times \vec{G}) = (\text{curl } \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\text{curl } \vec{G})$?

A.B. $\text{div}(\vec{F} \times \vec{G}) = (\text{div } \vec{F}) \times \vec{G} + \vec{F} \times (\text{div } \vec{G})$?

A.C. $\text{div}(\vec{F} \times \vec{G}) = (\text{curl } \vec{F}) \times \vec{G} + \vec{F} \times (\text{curl } \vec{G})$?

B. You're a farmer trying to maximize crop yield. If you add N units of nitrogen, P units of phosphorus, and K units of potassium to your farm field, then your yield is

$$Y = (16N + 4P + 8K)e^{-N-4P-2K}.$$

Your budget requires that $12N + P + K \leq 48$. Also, N, P, and K can't be negative.

B.A. Using techniques from Math 211, set up the problem in detail. Be clear about your strategy, what subproblems it produces, the role of each function, region, constraint, and equation, etc. (My goal here is to assess your knowledge of Math 211, not your skill in calculations from other courses. Your response should address all aspects of the problem that are relevant to Math 211 material. Your response should convince me that you understand which kind of problem this is, and which steps are required to solve problems of that kind, not just this specific problem. You might want to read part B before responding.)

B.B. Solve the problem from part A. (This part of the problem should consist entirely of material from other courses. If it doesn't, then your response to part A is inadequate. In fact, part B is worth zero points. I put it here, because some students may omit crucial information from their responses to part A. For those students, I might have to grade part B, to figure out what their grades for part A really are. So you might want to complete as much of part B as is practical.)

C. In $\mathbb{R}^3$, let $\vec{F} = \langle y\cos(xy), x\cos(xy) + z, y\rangle$. Find all potential functions for $\vec{F}$ (or show that none exist).

D. In $\mathbb{R}^2$, consider the work W done by the force field $\vec{F} = \langle x, y \rangle$ on a particle that moves from $(0, 0)$ to $(1, 1)$ along the parabola $y = x^2$.

D.A. Draw a picture. Does it suggest that $W > 0$, $W = 0$, or $W < 0$? Or not? Explain.

D.B. Compute the work.

E. In $\mathbb{R}^2$, compute the work W done by the force field $\vec{F} = \langle x^3, 3x^2 y \rangle$ on a particle that moves (along straight line segments) from $(1, 1)$ to $(2, 1)$ to $(1, 2)$ to $(1, 1)$.

F. On each TRUE-FALSE question below, there are four valid answers. If the correct answer is TRUE, then TRUE earns 3 points, TRUISH earns 2 points, FALSISH earns 1 point, and FALSE earns 0 points. If the correct answer is FALSE, then these point values are of course reversed. Do not write just T or F; write your answer completely and clearly. No explanation is needed.

F.A. For all smooth 3D $\vec{F}$, if curl $\vec{F} = \vec{0}$, then there is an f such that $\vec{F} = \text{grad } f$.

F.B. For all smooth 3D $\vec{F}$, if there is an f such that $\vec{F} = \text{grad } f$, then curl $\vec{F} = \vec{0}$.

F.C. For all smooth 3D $\vec{F}$, if div $\vec{F} = 0$, then there is an f such that $\vec{F} = \text{grad } f$.

F.D. For all smooth 3D $\vec{F}$, if there is an f such that $\vec{F} = \text{grad } f$, then div $\vec{F} = 0$.

F.E. For all smooth 3D $\vec{F}$, if div $\vec{F} = 0$, then there is a $\vec{G}$ such that $\vec{F} = \text{curl } \vec{G}$.

F.F. For all smooth 3D $\vec{F}$, if there is a $\vec{G}$ such that $\vec{F} = \text{curl } \vec{G}$, then div $\vec{F} = 0$.

F.G. For all smooth $\vec{F}$ that are defined on all of $\mathbb{R}^3$, if there is an f such that $\vec{F} = \text{grad } f$, then all line integrals of $\vec{F}$ are path-independent.

F.H. For all smooth $\vec{F}$ that are defined on all of $\mathbb{R}^3$, if all line integrals of $\vec{F}$ are path-independent, then there is an f such that $\vec{F} = \text{grad } f$.