A.A. Yes, this guess is correct, as we can check algebraically:

$$\begin{aligned}
\operatorname{div}(\vec{F} \times \vec{G}) &= \operatorname{div}\langle F_2G_3 - F_3G_2, F_3G_1 - F_1G_3, F_1G_2 - F_2G_1\rangle \\
&= (F_2G_3 - F_3G_2)_x + (F_3G_1 - F_1G_3)_y + (F_1G_2 - F_2G_1)_z \\
&= F_{2x}G_3 + F_2G_{3x} - F_{3x}G_2 - F_3G_{2x} \\
&\quad + F_{3y}G_1 + F_3G_{1y} - F_{1y}G_3 - F_1G_{3y} \\
&\quad + F_{1z}G_2 + F_1G_{2z} - F_{2z}G_1 - F_2G_{1z} \\
&= (F_{3y} - F_{2z})G_1 + (F_{1z} - F_{3x})G_2 + (F_{2x} - F_{1y})G_3 \\
&\quad + F_1(G_{2z} - G_{3y}) + F_2(G_{3x} - G_{1z}) + F_3(G_{1y} - G_{2x}) \\
&= (\operatorname{curl} \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\operatorname{curl} \vec{G}).
\end{aligned}$$

A.B. This guess is incorrect, because on the right side it is non-sensical to write the cross product of the scalar field div $\vec{F}$ with the vector field $\vec{G}$.

A.C. This guess is incorrect, because it is non-sensical to equate the scalar field on the left side with the vector field on the right side.

B.A. We want to maximize $Y(N, P, K)$ on the region R defined by $N \geq 0$, $P \geq 0$, $K \geq 0$, and $12N + P + K \leq 48$. This R is closed and bounded, and Y is continuous on R, so the maximum is guaranteed to exist among the critical points of Y and the boundary points of R.

The gradient of Y is

$$\nabla Y = e^{-N-4P-2K}\langle 16 - 16N - 4P - 8K, 4 - 64N - 16P - 32K, 8 - 32N - 8P - 16K\rangle.$$

The critical points are where ∇Y is undefined (which never happens) and where $\nabla Y = \vec{0}$. So the critical points require us to check all points (N, P, K) such that

$$\begin{aligned}
16N + 4P + 8K &= 16, \\
64N + 16P + 32K &= 4, \\
32N + 8P + 16K &= 8.
\end{aligned}$$

That's a linear algebra problem, not a Math 211 problem. [Incidentally, there are no such points, as you can see by comparing the second equation to twice the third equation.]

The boundary requires us to check four triangles of points. Three of the triangles are where $N = 0$, $P = 0$, or $K = 0$. On those three triangles, $Y = 0$, which is certainly not the maximum (because Y is positive when N, P, K are all slightly positive). So it remains to check the fourth boundary triangle, where $K = 48 - 12N - P$. So we need to optimize a new function

$$Z(N, P) = Y(N, P, 48 - 12N - P)$$

on the region of the N-P-plane defined by $N \geq 0$, $P \geq 0$, and $48 - 12N - P \geq 0$. This is a similar optimization problem — but of one lower dimension. The gradient ∇Z is never undefined, but we need to check points where it's zero. There are three boundary line segments, but we don't need to check the ones where $N = 0$ or $P = 0$. We need to check where $48 - 12N - P = 0$. To do that, we define a new function

$$W(N) = Z(N, 48 - 12N)$$

to be optimized for N in the interval $[0, 48 - 12N]$. This is a Calculus 1 problem.

[For the record, I graded this as a 12-point problem, assigning one point for each of the following aspects:

1. We want to maximize $Y(N, P, K)$
2. on the region R defined by $N \geq 0$, $P \geq 0$, $K \geq 0$, and
3. $12N + P + K \leq 48$.
4. This Y is continuous on R,
5. and R is closed and bounded,
6. so the max is guaranteed to occur at the boundary points of R or
7. at the critical points of Y, which are where
8. ∇Y is undefined or
9. $\nabla Y = \vec{0}$.
10. The gradient of Y is $\nabla Y = \ldots$.
11. The $N = 0$, $P = 0$, and $K = 0$ boundary triangles can't have the max.
12. So let's optimize Y restricted to the boundary triangle where $12N + P + K = 48$.

I hope that this rubric helps you see what I value greatly (understanding what the method is and why we follow it) and what I value less (mindless calculations).]

B.B. [I'll omit the solution. It's long, and my response to part A already demonstrates that I understand the Math 211 content of the problem, I hope.]

C. [We could check that $\partial P/\partial y = \partial Q/\partial x$, $\partial Q/\partial z = \partial R/\partial y$, and $\partial P/\partial z = \partial R/\partial x$. Those equations do hold, which means that we can't immediately rule out the existence of a potential function. But let's jump straight to trying to find a potential function.] If a potential function f exists, then

$$f = \int P\,dx = \int y \cos(xy)\,dx = \sin(xy) + g(y, z)$$

for some function g. Then

$$x \cos(xy) + z = Q = \partial f/\partial y = x \cos(xy) + \partial g/\partial y,$$

which implies that $g(x, y) = yz + h(z)$ for some function h. Then

$$y = R = \partial f/\partial z = \partial g/\partial z = y + h'(z),$$

which implies that $h = C$ is constant. We conclude that

$$f = \sin(xy) + yz + C$$

is a potential function for $\vec{F}$, for any constant C. Moreover, because potential functions are unique up to additive constants, we have found all potential functions for $\vec{F}$.

D.A. [I'll omit the picture in these typed solutions. You should not.] The picture shows that $\vec{F}$ sympathizes with the velocity vectors on the curve. That is, the dot products that get integrated into work are positive rather than negative or zero. Therefore the work should be positive.

D.B. Let C be the curve in question. Notice that $\vec{F} = \text{grad } f$, where $f(x, y) = x^2/2 + y^2/2$. So, by the fundamental theorem of calculus for line integrals, the work is

$$\int_C \vec{F} \cdot d\vec{x} = f(1, 1) - f(0, 0) = 1.$$

[Alternatively, here's the brute-force approach. Parametrize C as $\vec{x}(t) = (t, t^2)$ for $0 \leq t \leq 1$. So $\vec{x}\,'(t) = \langle 1, 2t \rangle$, and the work is

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{x} &= \int_0^1 \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt \\ &= \int_0^1 \langle t, t^2 \rangle \cdot \langle 1, 2t \rangle\, dt \\ &= \int_0^1 t + 2t^3\, dt \\ &= \left[t^2/2 + t^4/2\right]_0^1 \\ &= 1. \end{aligned}$$

That's not terribly long, but using the fundamental theorem is quicker.]

E. [Draw a picture!] The curve in question is a simple closed curve oriented counter-clockwise. Let D be the region enclosed, and let P, Q be the components of $\vec{F}$. Then Green's theorem says that the work is

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{x} &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dy \, dx \\
&= \int_1^2 \int_1^{3-x} 6xy \, dy \, dx \\
&= \int_1^2 \left[3xy^2\right]_{y=1}^{3-x} \, dx \\
&= \int_1^2 3x(9 - 6x + x^2) - 3x \, dx \\
&= \int_1^2 24x - 18x^2 + 3x^3 \, dx \\
&= \left[12x^2 - 6x^3 + 3x^4/4\right]_1^2 \\
&= (48 - 48 + 12) - (12 - 6 + 3/4) \\
&= 6 - 3/4.
\end{aligned}$$

[Alternatively, the brute-force approach requires parametrizing the three boundary curves and adding the three corresponding work integrals. I'd rather not do it.]

F.A. FALSE. [A counter-example is given in our `gradCurlDiv.pdf` handout.]

F.B. TRUE. [This is just saying that curl(grad f) $= \vec{0}$ for any smooth f.]

F.C. FALSE. [I think that we have never studied this issue specifically, but here's a simple example: $\vec{F} = \langle x^2, -2xy, 0\rangle$.]

F.D. FALSE. [I think that we have never studied this issue specifically, but the statement is essentially claiming that all f are harmonic, which is not true. Consider $f = x^2$, which leads to $\vec{F} = \langle 2x, 0, 0\rangle$.]

F.E. FALSE. [A counter-example is given in our `gradCurlDiv.pdf` handout.]

F.F. TRUE. [This is just saying that div(curl $\vec{G}$) $= 0$ for any smooth $\vec{G}$.]

F.G. TRUE. [If $\vec{F}$ is conservative, then $\vec{F}$ has path-independent line integrals.]

F.H. TRUE. [If $\vec{F}$ has (only) path-independent line integrals, then $\vec{F}$ is conservative.]