You have 60 minutes.

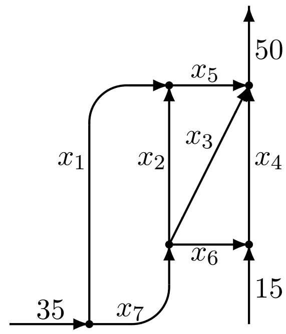
No notes, books, calculators, computers, etc. are allowed.

Show all of your work, in as organized a manner as possible. Incorrect answers with solid work often earn partial credit. Correct answers without explanatory work rarely earn full credit.

Perform as much algebraic simplification as you can. Simple correct answers are generally preferred over complicated correct answers. Do simple arithmetic, but don't bother to do complicated arithmetic. Mark your final answer clearly.

Good luck. :)

You are studying the Amazon River delta (satellite photo below left, NASA, 2005). As the river flows generally northeast, it repeatedly splits and merges to form a complicated pattern of channels. You can estimate the flux (volume per second) in a channel using a device called a stream gauge. At any merge or split, the total flux should be conserved; for example, two channels of fluxes 11 and 20 could merge to form a channel of flux 31, which could then split into channels of fluxes 23, 3, and 5. So far, you have modeled part of the delta as the diagram below right, with some fluxes already measured and other fluxes still unknown.



A.A. Write, but don't solve, a system of linear equations for the unknown fluxes.

A.B. Without solving the system, make a guess: Do you have enough information to find the unknown fluxes? If not, then what is the cheapest way to get enough information? Explain.

B. Solve the following system of equations by fully row-reducing an augmented matrix. Your answer should make clear, how many solutions there are, and (if any) what they are.

$$\begin{aligned} 3x - 6y + 12z &= 2, \\ 2x + z &= 1, \\ -x - 6y + 10z &= 0. \end{aligned}$$

C.A. Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix}$. (By the way, $\pi/4 = 45°$, and the sine and cosine of $\pi/4$ are both $1/\sqrt{2}$.) Make five sketches: an untransformed object, and that object transformed by A, by B, by AB, and by BA. Your untransformed object should have enough asymmetry that your sketches convey the effects of the transformations clearly.

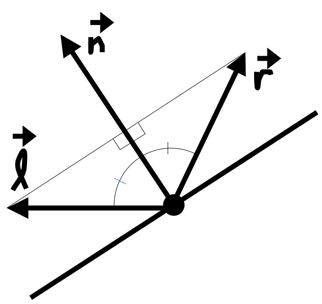
C.B. Does A^{-1} exist? If so, what is it? Does B^{-1} exist? If so, what is it?

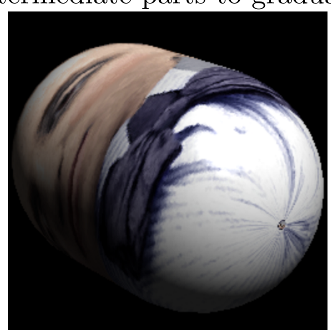
D.A. Write a matrix Q, in terms of A and/or $\vec{b}$, such that the least-squares solution to $A\vec{x} = \vec{b}$ is the actual solution to $A\vec{x} = Q\vec{b}$.

D.B. Compute the least-squares solution in the example where $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Much of computer graphics is about how light bounces off surfaces. Recall the diagram below left from homework. A triangle with normal $\vec{n}$ is being viewed edge-on, so that it looks like a line segment. The vector $\vec{\ell}$ points toward a light source. The vector $\vec{r}$ is not relevant to today's problem; ignore it. You may assume that both $\vec{n}$ and $\vec{\ell}$ are unit vectors.

When light hits a surface that isn't perfectly smooth, the microscopic texture of the surface causes reflected light to scatter in all directions. If the light hits the surface perpendicularly, then the intensity (a scalar) of the scattered light is nearly as great as the intensity of the incoming light. If the light arrives nearly parallel to the surface, so that it "glances", then the scattered intensity is nearly zero. At intermediate angles, the scattered intensity takes on intermediate values smoothly. In the example below right, a capsule made of hundreds of small triangles is being lit from above-right. Scattering causes the part facing the light to appear bright, the part facing away from the light to appear dark, and the intermediate parts to gradually fade.





E. Invent a formula for the scattered light intensity, in terms of $\vec{n}$ and $\vec{\ell}$. There is no one right answer, but some answers are better than others. The best answers use material from this course. The best answers are computationally fast, in that they use only additions, subtractions, and/or multiplications of the numbers that make up $\vec{n}$ and $\vec{\ell}$.