A.A. This problem strongly resembles the problems that we've done about electrical flow in circuits and traffic flow on roads. Each junction (merge or split) yields an equation involving the fluxes at that junction. So we have

$$\begin{aligned} x_1 + x_7 &= 35, \\ x_1 + x_2 - x_5 &= 0, \\ x_2 + x_3 + x_6 - x_7 &= 0, \\ x_3 + x_4 + x_5 &= 50, \\ x_4 - x_6 &= 15. \end{aligned}$$

A.B. There are five equations in seven unknowns. If the equations happen to contradict each other, then there is no solution. But that's improbable. My guess is that we'll end up with two free variables and hence infinitely many solutions. To really nail down the fluxes, we must measure two of the unknown fluxes, so that only five unknowns remain.

(Bonus question: It might seem that measuring one well-chosen flux could be enough. For example, if we measured x_1, then wouldn't we know x_7 too? Yes, but why is that inadequate?)

(Bonus question: How could I tweak the original problem, to make the equations contradict each other? There is a simple answer.)

B. After a few steps, which I'll omit in these solutions, I think that we get down to the fully reduced augmented matrix

$$\left[\begin{array}{cccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{21}{12} & -\frac{1}{12} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So z is a free variable, and there are infinitely many solutions. For any choice of value for z, we get $x = \frac{1}{2} - \frac{1}{2}z$ and $y = -\frac{1}{12} + \frac{21}{12}z$.

C.A. I'll not draw the pictures in these solutions, but here is a verbal description. Your A picture should show your object smashed onto the y-axis. Your B picture should show your object rotated through an angle of $\pi/4$ counterclockwise about the origin. Your AB picture should be rotated then smashed. Your BA picture should be smashed then rotated.

C.B. No, A^{-1} does not exist, because $A_{11}A_{22} - A_{12}A_{21} = 0$. This makes geometric sense, because A smashes the plane onto the y-axis, and that transformation is not invertible. Meanwhile, B is a rotation matrix, so it's orthogonal, so its inverse is just its transpose:

$$B^{-1} = B^\top = \left[\begin{array}{cc} \cos \pi/4 & \sin \pi/4 \\ -\sin \pi/4 & \cos \pi/4 \end{array}\right].$$

(Notice that

$$B^{-1} = \begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix}.$$

This answer makes sense; B rotates by $\pi/4$, so B^{-1} should rotate by $-\pi/4$. But this observation requires a level of trigonometry knowledge that I don't expect all students to have.)

D.A. As we have studied in class, Q is the projection onto the column space of A, and we should have $Q = A(A^\top A)^{-1}A^\top$, if $A^\top A$ is invertible.

D.B. To compute the least-squares solution, we could compute $Q\vec{b}$ and then solve $A\vec{x} = Q\vec{b}$, but it is faster to use the fact (which we have seen in class) that the least-squares solution is

$$\begin{aligned}
\vec{x} &= (A^\top A)^{-1}A^\top \vec{b} \\
&= \left(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 1 \\ 1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \frac{1}{17} \begin{bmatrix} 6 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \frac{1}{17} \begin{bmatrix} 9 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} 9/17 \\ 7/17 \end{bmatrix}.
\end{aligned}$$

E. A great answer is that the scattered intensity is $\vec{n} \cdot \vec{\ell}$ times the original intensity. Why? Because $\vec{n} \cdot \vec{\ell} = 1$ when the light hits the surface perpendicularly, and $\vec{n} \cdot \vec{\ell} = 0$ when the light glances, and in general $\vec{n} \cdot \vec{\ell} = \cos\theta$, where θ is the angle between $\vec{n}$ and $\vec{\ell}$. Moreover,

$$\vec{n} \cdot \vec{\ell} = n_1\ell_1 + n_2\ell_2 + n_3\ell_3,$$

so the computation is fast. (In fact, this is the standard answer used in computer graphics. It's called Lambertian diffuse reflection. It was invented by Lambert around 1760.)