A. We are trying to solve the system of equations

$$\begin{aligned} -2m + b &= 0, \\ 0m + b &= -1, \\ 1m + b &= 1, \\ 2m + b &= -1. \end{aligned}$$

This system is of the form $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

There is no solution (because the four points do not lie on one line), but we can obtain the least-squares solution as

$$\begin{aligned} \begin{bmatrix} m \\ b \end{bmatrix} &= (A^\top A)^{-1} A^\top \vec{b} \\ &= \left(\begin{bmatrix} -2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &= \frac{1}{35} \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &= \frac{1}{35} \begin{bmatrix} -3 \\ -8 \end{bmatrix} \\ &= \begin{bmatrix} -3/35 \\ -8/35 \end{bmatrix}. \end{aligned}$$

Therefore the best-fit line is $y = -\frac{3}{35}x - \frac{8}{35}$.

B.A. I multiply the received message $\vec{v} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}^\top$ by the checker matrix C (the matrix on the right in the statement of the problem) to produce $C\vec{v} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. This is the first column of the checker matrix, or equivalently 1 in binary. It tells us that an error occurred in the first bit of the message. So the corrected message is $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}^\top$.

If we multiply it by the decoder matrix D (the matrix in the middle), or equivalently just yank out the first four bits, we find that Tagak's original message was $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\top$ or 0100.

B.B. I again compute the error message $C\vec{v} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. Because it is not zero, I know that at least one error has occurred. But I cannot correct the error. So I ask Tagak to re-send the message.

C.A. Set $\vec{x} = \vec{v} - \vec{w}$. Then $\vec{x} \neq \vec{0}$, and

$$A\vec{x} = A(\vec{v} - \vec{w}) = A\vec{v} - A\vec{w} = \vec{0}.$$

So we see that $A\vec{x} = \vec{0}$ has a non-trivial solution.

The converse is also true: If $A\vec{x} = \vec{0}$ has a non-trivial solution $\vec{x}$, then setting $\vec{v} = \vec{x}$ and $\vec{w} = \vec{0}$ shows that there are two distinct vectors $\vec{v}$ and $\vec{w}$ such that $A\vec{v} = A\vec{w}$.

C.B. Well,

$$\det \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 2 & 5 & 2 \end{bmatrix} = 1 \det \begin{bmatrix} 1 & 2 \\ 5 & 2 \end{bmatrix} - 2 \det \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} = 1(-8) - 2(-6) = 4.$$

C.C. Problem C.B tells us that B is invertible. So if $B\vec{v} = B\vec{w}$, then multiplying both sides of the equation (on the left) by B^{-1} gives $\vec{v} = \vec{w}$. So no, there cannot be distinct $\vec{v}$, $\vec{w}$ such that $B\vec{v} = B\vec{w}$. [If this conclusion is surprising, given the setup in Problems C.A and C.B, it's because I mistakenly computed $\det B = 0$ when I was writing the exam.]

D. The characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & 2 \\ 3 & 1-\lambda \end{bmatrix} \\ &= (2-\lambda)(1-\lambda) - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1). \end{aligned}$$

So the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$. To find an eigenvector for λ_1, we solve $(A - 4I)\vec{x} = \vec{0}$:

$$\begin{bmatrix} -2 & 2 & 0 \\ 3 & -3 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $\vec{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ is an eigenvector for λ_1. To find an eigenvector for λ_2, we solve $(A + I)\vec{x} = \vec{0}$:

$$\begin{bmatrix} 3 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $\vec{v}_2 = \begin{bmatrix} -2/3 & 1 \end{bmatrix}^\top$ is an eigenvector for λ_2. Putting it all together, we have $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & -2/3 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

E. We are asked to find a basis for the space of solutions to $A\vec{x} = \vec{0}$. Notice that the rows of A are all scalar multiples of each other. So the reduced form of the augmented matrix for $A\vec{x} = \vec{0}$ is quickly found to be

$$\begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For solutions $\vec{x} = \begin{bmatrix} x & y & z \end{bmatrix}^\top$, this matrix shows that x is free, $y = 2z$, and z is free. Because there are two free variables, the solution space is two-dimensional, and our basis should consist of two vectors. Here are two simple vectors that fit the requirements:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$