Please write your name at the top of this cover page and nowhere else.

A crib sheet is permitted according to our agreed-upon rules (both sides of one piece of paper, etc.). Otherwise, no notes, books, phones, calculators, computers, etc. are allowed.

If you cannot understand what a question is asking, then ask for clarification. If you cannot obtain clarification, then include your interpretation of the problem in your solution. Never interpret a problem in a way that renders it trivial.

Show all of your work, in as organized a manner as possible. Incorrect answers with solid work often earn partial credit. Correct answers without explanatory work rarely earn full credit.

Perform as much algebraic simplification as you can. Simple correct answers are generally preferred over complicated correct answers. Mark your final answer clearly.

You have 150 minutes (two and a half hours).

Good luck. :)

Suppose that A is a non-invertible $n \times n$ matrix. What can you tell me about ...

A.A. $\det A$?

A.B. the column space of A?

A.C. the SVD of A?

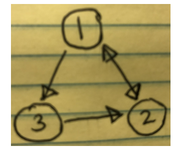
A.D. $A^\top$?

You are studying a species that has two notable life stages: juvenile and adult. The juvenile stage lasts one year; after that, the juvenile is an adult, ready to reproduce, if it has survived. Each year, each adult female produces, on average, 1/2 of a juvenile female. Each year, 1/5 of the juvenile females survive to become adult females, and 9/10 of adult females survive.

B.A. Describe this situation as a discrete dynamical system. Be explicit about what vector $\vec{x}$ is changing from year to year, and what matrix A is changing $\vec{x}$.

B.B. Describe the long-term trend of the system. Does the species go extinct? Does it thrive and expand? What fraction of females are juvenile, and what fraction are adult?

The figure below right shows a tiny World Wide Web consisting of just three web pages. The first page links to the other two. The second page links to the first. The third page links to the second. If a web crawler is at a page at time k, then at time $k+1$ it has moved to one of that page's linked pages (randomly with equal probability).



C.A. Describe the situation as a Markov chain $\vec{x}_{k+1} = P\vec{x}_k$.

C.B. Assume that P^5 is entirely positive. What is the steady state of the Markov chain?

C.C. Describe the meaning of that steady state in English.

D. The matrix below left row-reduces to the matrix below right. Find a basis for the null space of the matrix below left.

$$\begin{bmatrix} 0 & 3 & 6 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

E. Compute the SVD of $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$.

In principal component analysis, we transform data points $\vec{x}$ to other vectors $\vec{w}$ that are easier to study in some ways. For this problem, suppose that a data set of d-dimensional vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ has sample mean $\vec{m}$ and sample covariance $S = PDP^\top$, where P is orthogonal and D is diagonal.

F.A. What does the first column of P tell us about the data set $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$?

F.B. What is $\vec{w}_{37}$, in terms of $\vec{x}_{37}$ and the information above?

F.C. How do $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ help us visualize the data set? What do we do with them?