Recently in class we proved that ACC_{TM} is not decidable. The proof is confusing and ingenious, and students may wonder how it was ever discovered. But it's less surprising, if you know about the history of logic leading up to it. In this era, philosophers and mathematicians often made arguments using self-reference. Here are a few paradoxes that arise from self-reference.

1 Barber paradox

Imagine a small town with a single barber. Some people in this town cut their own hair. Everyone else gets their hair cut by the barber. That is, the barber cuts the hair of every person who doesn't cut their own hair, and only those people. Make sense?

So who cuts the barber's hair?

2 Russell's paradox

The following paradox is much like the barber paradox, but dressed up to seem less frivolous. Actually, it's historically important in clarifying the rules by which sets can be constructed.

Consider all possible sets. Let S be the set of all sets that do not contain themselves as elements. That is,

$$S = \{T : T \notin T\}.$$

Is $S \in S$?

3 Typewriter paradox

You are given a typewriter and a sheet of paper. (If you prefer, replace the typewriter with a computer and a printer, but use a fixed-width typeface such as Courier.) Your job is to type a number on the sheet. How many different numbers can you type?

For the sake of argument, suppose that you can type 80 characters per row and 40 rows per sheet. So you can type 3200 digits. So you can type numbers from 0 to $999\ldots999$ (with 3200 nines in there). That's 10^{3200} different numbers. Not bad.

But if you allow yourself to use symbols other than digits, then you can type even more numbers. For example, you can type

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2^999999999999999999999999999999999999999999999999999999999999999999
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which is much greater than any of the numbers mentioned above. You can type

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Let -> denote Knuth's up-arrow operator. Then the number that I wish
to type is 2 -> 999999999999999999999999999999999999999999999999999.
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That number is even greater. And here's a small number that we haven't mentioned yet:

`the natural logarithm of the square root of 5`

And we can keep going like this, imagining more and more sophisticated ways to type more and more numbers.

So is there any bound on the number of numbers that you can type? Well, suppose that your typewriter has 100 symbols that it can produce: 26 lower-case letters, 26 upper-case letters, various punctuation marks, white space, etc. Then there are 100^{3200} different pages that you can type. Many of these will be nonsense, but some will be numbers. In any event, 100^{3200} is an upper bound on how many numbers you can type. And even if you disagree with the specific parameter values 80, 40, and 100 that I've picked, picking any other values leads to a similar bound. In any event, there are only finitely many numbers that can be typed. Hence there is a greatest number that can be typed. I don't know what it is, off the top of my head, but this greatest number definitely exists. Right?

So what if you type the following?

`one more than the greatest number that can be typed on this sheet`

4 Curry's paradox

The big assumption behind proof by contradiction is that our logical system is fundamentally self-consistent, with no contradictions. If any contradiction existed — meaning, a statement C such that C is true and C is false — then that contradiction could be used to prove *anything*. To see how, let A be any statement. For the sake of contradiction, assume that A is false. We know that C is true. But we also know that C is false. This contradiction implies that the original assumption is false. Thus A is true. Yikes! So contradictions must be avoided.

Now consider the statement "If this statement is true, then $5 = 3$." This statement is of the form $P \to Q$, where P is "this statement is true" and Q is "$5 = 3$". For the statement to avoid self-contradiction, P and $P \to Q$ must have the same truth value. The truth table for an implication $P \to Q$ is

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

The only row of this truth table, where P and $P \to Q$ have the same truth value, is the last row. In this row, Q is true. Thus $5 = 3$.