There's a lot of algebra and linear algebra on this homework. If you get stuck on the linear algebra, then you might find the list of facts on the back of this page helpful. :)

Let $\mathbb{R}^{m \times n}$ be the vector space of $m \times n$ matrices. The *Frobenius inner product* on $\mathbb{R}^{m \times n}$ is defined by

$$\langle A, B \rangle = \operatorname{tr}\left(A^\top B\right).$$

When $n = 1$, this inner product is the dot product on $\mathbb{R}^m$.

A. Prove that the Frobenius inner product is indeed an inner product on $\mathbb{R}^{m \times n}$.

In class I meant to claim (before I ran out of time) that an inner product $\langle \ldots, \ldots \rangle$ on a vector space V induces a norm $\|\ldots\|$ on V by $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.

B. Prove this claim. (This exercise is drastically shorter and easier, if you cite a fact about inner products that was mentioned in class.)

Let G be a positive-definite, symmetric $n \times n$ matrix. As we discussed in class, $\langle \vec{v}, \vec{w} \rangle = \vec{v}^\top G \vec{w}$ defines an inner product on $\mathbb{R}^n$. The dot product is the case where $G = I$. Let $\|\ldots\|_G$ denote the norm arising from this inner product. Notice that $\|\ldots\|_I$ is the standard norm on $\mathbb{R}^n$.

C.A. Let m and M be the least and greatest eigenvalues of G, respectively. Prove that

$$\sqrt{m}\, \|\vec{v}\|_I \leq \|\vec{v}\|_G \leq \sqrt{M}\, \|\vec{v}\|_I$$

(for all $\vec{v} \in \mathbb{R}^n$). (This is probably the most difficult part of this homework.)

C.B. Prove that, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|\vec{v}\|_I < \delta \Rightarrow \|\vec{v}\|_G < \epsilon$.

C.C. Prove that, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|\vec{v}\|_G < \delta \Rightarrow \|\vec{v}\|_I < \epsilon$.

Epilogue: Not today, but soon, we will be able to prove an interesting result based on this exercise: All values of G generate the standard topology on $\mathbb{R}^n$. In other words, different values of G produce different geometries but the same topology. This result is fairly typical. Topology is "geometry with much of the detail ignored".

Here are some of the definitions and theorems about linear algebra over $\mathbb{R}$, that I found useful in solving these problems. You may assume them without proof.

The definition of matrix multiplication: If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$, and $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$.

The definition of the trace: If A is $n \times n$, then $\mathrm{tr} A = \sum_{i=1}^{n} A_{ii}$.

About transposition: $(AB)^\top = B^\top A^\top$, and $\mathrm{tr}\left(A^\top\right) = \mathrm{tr} A$.

About orthogonality: An $n \times n$ matrix Q is orthogonal if and only if (Q is invertible and) $Q^{-1} = Q^\top$. If Q is orthogonal, then it preserves the standard norm on $\mathbb{R}^n$, meaning that $\|Q\vec{v}\| = \|\vec{v}\|$ for all $\vec{v}$.

Spectral theorem: If G is a symmetric $n \times n$ matrix, then G is diagonalizable, has only real eigenvalues, and has perpendicular eigenvectors. That is, there exist real matrices Q and D such that $G = Q^{-1} D Q$, D is diagonal, and Q is orthogonal.

The definition of positive-definiteness: A symmetric $n \times n$ matrix G is positive-definite if all of its eigenvalues are positive.