In class I claimed that a norm on a vector space V induces a metric on the set V by $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$.

A. Prove this claim.

Near the start of Section 21, Munkres claims: If (X, d) is a metric space and $Y \subseteq X$, then the restriction of d to $Y \times Y$ is a metric on Y, and the topology that it induces on Y matches the subspace topology on Y.

B. Prove the second part of this claim, that the topologies match. (You may assume the first part of the claim, that the metric restricts to a metric, which is easier to prove.)

For all positive-definite, symmetric $n \times n$ matrices G, we know that $\langle \vec{v}, \vec{w} \rangle_G = \vec{v}^\top G \vec{w}$ is an inner product on $\mathbb{R}^n$ and $\|\vec{v}\|_G = \sqrt{\langle \vec{v}, \vec{v} \rangle_G}$ is the norm induced by that inner product. We also know, from previous homework, that

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \forall \vec{v} \in \mathbb{R}^n \quad \|\vec{v}\|_I < \delta \Rightarrow \|\vec{v}\|_G < \epsilon$$

and

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \forall \vec{v} \in \mathbb{R}^n \quad \|\vec{v}\|_G < \delta \Rightarrow \|\vec{v}\|_I < \epsilon.$$

That previous homework foreshadowed the following result, which we are now ready to prove.

C. All values of G induce the same topology on $\mathbb{R}^n$. Prove so. (Hint: Lemma 13.3.)

D. Let X_T be a metrizable space with only finitely many points. Prove that T is the discrete topology on X. (Hint: If it helps, start with $|X| = 2$?)