Let $\mathbb{R}^{n\times n}$ be the set of real $n \times n$ matrices. If you start with an $n \times n$ matrix, and list its entries column-by-column (or row-by-row, or in any other consistent order), then you obtain an (n^2)-dimensional vector. In this way, $\mathbb{R}^{n\times n}$ is bijectively identified with $\mathbb{R}^{n^2}$. Moreover, the Frobenius inner product on $\mathbb{R}^{n\times n}$ (Day 05 Homework) is identified with the dot product on $\mathbb{R}^{n^2}$. Hence the induced norm $\|\ldots\|$ on $\mathbb{R}^{n\times n}$ matches the standard norm on $\mathbb{R}^{n^2}$, and the induced metrics and topologies match too. Everything is pretty simple.

Let $\mathrm{SO}(n) \subseteq \mathbb{R}^{n\times n}$ be the set of matrices R such that $R^\top R = I$ and $\det R = 1$. Under the usual identification of matrices with linear transformations (with respect to the standard basis of $\mathbb{R}^n$), $\mathrm{SO}(n)$ is the set of rotations of $\mathbb{R}^n$. By the way, it's a group.

A.A. Prove that $\mathrm{SO}(n)$ is a closed subset of $\mathbb{R}^{n\times n}$. (This is a medium-length exercise. Hint: Show that each of the $n^2 + 1$ scalar equations, that define $\mathrm{SO}(n)$, defines a closed subset.)

A.B. Prove that, for all $R \in \mathrm{SO}(n)$, $\|R\| \leq \sqrt{n}$. (This is a short exercise.)

A.C. Prove that $\mathrm{SO}(n)$ is compact. (This is a short exercise.)

Because we seem to live in $\mathbb{R}^3$, $\mathrm{SO}(3)$ is the $\mathrm{SO}(n)$ that shows up most often scientific applications: spacecraft maneuvers, robotic control, computer vision, motions of tectonic plates across Earth's surface, medical imaging, etc. A peculiar feature of $\mathrm{SO}(3)$ is its relationship to another structure called the *quaternions*. They take a while to develop well, so let's not. All you need to know right now is that the quaternions are naturally viewed as $\mathbb{R}^4$, the unit quaternions are naturally viewed as $\mathbb{S}^3 \subseteq \mathbb{R}^4$, and there is a two-to-one covering map $p : \mathbb{S}^3 \to \mathrm{SO}(3)$ such that $p(x) = p(-x)$.

B. What is the fundamental group of $\mathrm{SO}(3)$? Your answer should be some easy-to-understand group that is isomorphic to the fundamental group. (I might have mentioned the answer in class, but you may not simply cite class. You should demonstrate that you know the reason why the answer is what it is, based on the information given above. This is a short exercise.)

By the way, problem B is intimately connected to the phenomenon of *spin* in physics.

C. Describe a system of plotting data, in which a data set of n rotations (of three-dimensional space) appears as a set of n points in a three-dimensional plot. What is the shape of that plot? (This is a medium-length exercise. Hint: Compare to the Day 22 homework.)